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ABSTRACT

This guide is primarily designed to help those who are concerned with improving mathematics education in Georgia schools. The format is intended to make planning a new mathematics curriculum easier at the local level and still leave designers free to develop courses which are best suited for their students. In addition to presenting an outline entitled, Steps for Developing a Secondary School Mathematics Curriculum, the document covers: Goals of Mathematical Learning; Problem Solving; Strategies for Mathematics Instruction; Evaluating Mathematics Learning--Topics, Objectives, and Courses; and Instructional Resources. The four appendices include: (A) Organizations for the Essentials of Education; (B) Position Paper on Basic Mathematical Skills of the National Council of Supervisors of Mathematics; (C) Recommendations for the Preparation of High School Students for College Mathematics Courses; and (D) Correlation of Georgia Statewide Basic Skills Test Indicator Clusters and Secondary School Mathematics Collection Objectives. The document concludes with a set of pamphlets on Careers in Mathematics, including teacher directions regarding duplication, distribution, and classroom use. (MP)

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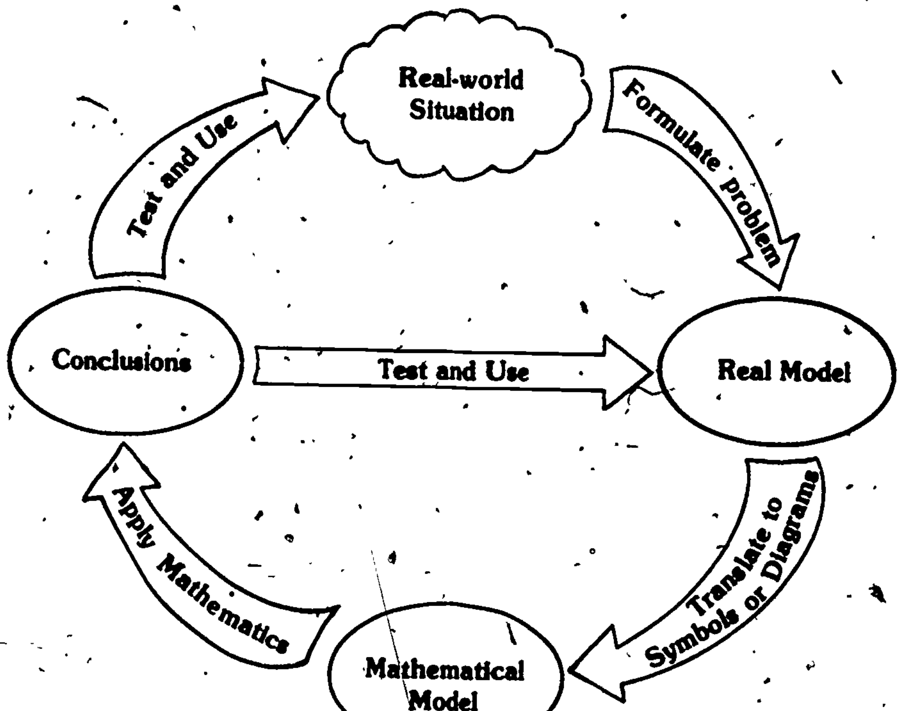
Mathematics for Georgia Secondary Schools



Division of Curriculum Services
Office of Instructional Services
Georgia Department of Education
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Charles McDaniel, State Superintendent of Schools
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Bill Bompart, Augusta College, Augusta
Pamela Brannen, Emory University, Atlanta, formerly with Fulton County Schools
Carol Dugger (deceased — formerly with Pike County, Zebulon)
John P. Downes, Georgia State University, Atlanta
Susan Fuller, Griffin Cooperative Educational Services Agency, Griffin
Thomas Hall, Mercer University in Atlanta, Atlanta
Hiram D. Johnston, Georgia State University, Atlanta
Ray Lamb, Atlanta City Schools, Atlanta
Carolyn Moore, Franklin County Schools, Carnesville
Edith Robinson, Educational Testing Services, Princeton, New Jersey (formerly with University of Georgia, Athens)
Earl Swank, Coastal Plains Cooperative Educational Services Agency, Valdosta
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Diana S. Brown, Cobb County Schools, Marietta
W. Jerry Carnes, The Westminster Schools, Atlanta
Don Caskey, Marist School, Atlanta
Pamela Drummond, Cobb County Schools, Marietta
Marjorie Economopoulos, Fulton County Schools, Atlanta
Wayne Godfrey, Cobb County Schools, Marietta, Georgia
Ena Gross, Virginia Commonwealth University, Richmond, Virginia (formerly with Fulton County Schools, Atlanta)
Chuck Hixon, Woodward Academy, College Park
Ron Hutcheson, DeKalb County Schools, Decatur
Dale Kilheffner, Armstrong College, Savannah
Mary Ann Byrne Lee, University of Southern Mississippi, Hattiesburg, Mississippi (formerly with University of Georgia, Athens)
Michael Mahaffey, University of Georgia, Athens
Betty O'Quinn, Fulton County Schools, Atlanta
James Overton, DeKalb County, Decatur

Sharon Sheddan, United Computing Services, Atlanta (formerly with Georgia State University)

Brooks Shelhorse, LaGrange College, LaGrange

Gwendolyn Shufelt, University of Missouri, Kansas City, Missouri (formerly with Fulton County Schools)

Martha Slater, The Arlington Schools, Atlanta

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Jerry Townsend, Augusta College, Augusta

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Thomas J. Cooney, University of Georgia, Athens

Iris Dayoub, Fort Knox, Kentucky

~~John P. Downes~~, Georgia State University, Atlanta

Tom Hall, Mercer University in Atlanta, Atlanta

Hiram D. Johnston, Georgia State University, Atlanta

Gwendolyn Shufelt, University of Missouri, Kansas City, Missouri (formerly with Fulton County Schools, Atlanta)

James Wilson, University of Georgia, Athens

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Careers in Mathematics

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Foreword

Mathematics education is continually being reviewed at every level to keep instruction responsive and current. Resulting changes in mathematics education are based on developments in theories of learning, findings regarding instruction in the classroom, public opinion and most importantly, the evolving process of mathematics itself.

The Georgia Department of Education appointed a committee to study various aspects of mathematics at the secondary school level and to prepare guidelines for mathematics curriculum planning and assessment. This guide, *Mathematics for Georgia Secondary Schools*, is the committee's final product.

We appreciate the time and effort the committee members spent and commend them for this excellent publication. We are confident the guidelines presented here will help improve mathematics education in the secondary schools throughout the state.

Charles McDaniel
State Superintendent of Schools

Introduction

The purpose of this guide, *Mathematics for Georgia Secondary Schools*, is to help those who are concerned with improving mathematics education in Georgia Schools. The format of the guide is intended to make planning a new mathematics curriculum easier at the local level yet leaving the designers free to develop courses which are best suited for their students.

The committee members and writers who developed this guide have drawn information from successful practices presently in use and significant trends in mathematics education literature recommended by consultants.

Designers of a mathematics curriculum for the 1980s must critically consider the target population for whom they are planning — their abilities, interests, present and future needs. They must provide a program for all students which is broad enough in scope to furnish students with a solid base of essential mathematics skills. The program should be long range, aimed at developing skills which will be useful for many years to come. The writers of this guide have attempted to provide direction and information to aid schools in equipping students with those skills in mathematics which will enable them to become productive members of society and to pursue their personal goals.

We hope that this guide will be useful in reviewing, strengthening or developing the local mathematics curriculum and will lead to improved instruction for students in Georgia schools.

Lucille G. Jordan
Associate State Superintendent
Office of Instructional Services

R. Scott Bradshaw, Director
Division of Curriculum Services

Developing a Secondary School Mathematics Curriculum Guide

A major goal of secondary schools is to prepare students to function in modern society. To fulfill this goal, members of the school system and representatives of the community must plan carefully. This curriculum guide, *Mathematics for Georgia Secondary Schools*, was designed to help those involved in planning a mathematics curriculum for a local school system, local school or individual classroom.

We assume that a local curriculum planning committee with members from all subject areas has been established and has given input to the leaders of mathematics curriculum development. The committee (or an individual teacher) planning or revising a mathematics curriculum should develop steps or tasks to follow and designate people responsible for the tasks with associated time frames. A curriculum cannot be developed quickly. Adequate time must be given for study and discussion of trends and issues in mathematics, ways students learn and strategies of teaching.

Suggested steps for developing a secondary school mathematics curriculum are presented on the following pages. Some of these tasks may need revising to accommodate the local situation. Time frames are not included since time needed for these tasks will vary from school system to school system.

Steps for Developing a Secondary School Mathematics Curriculum

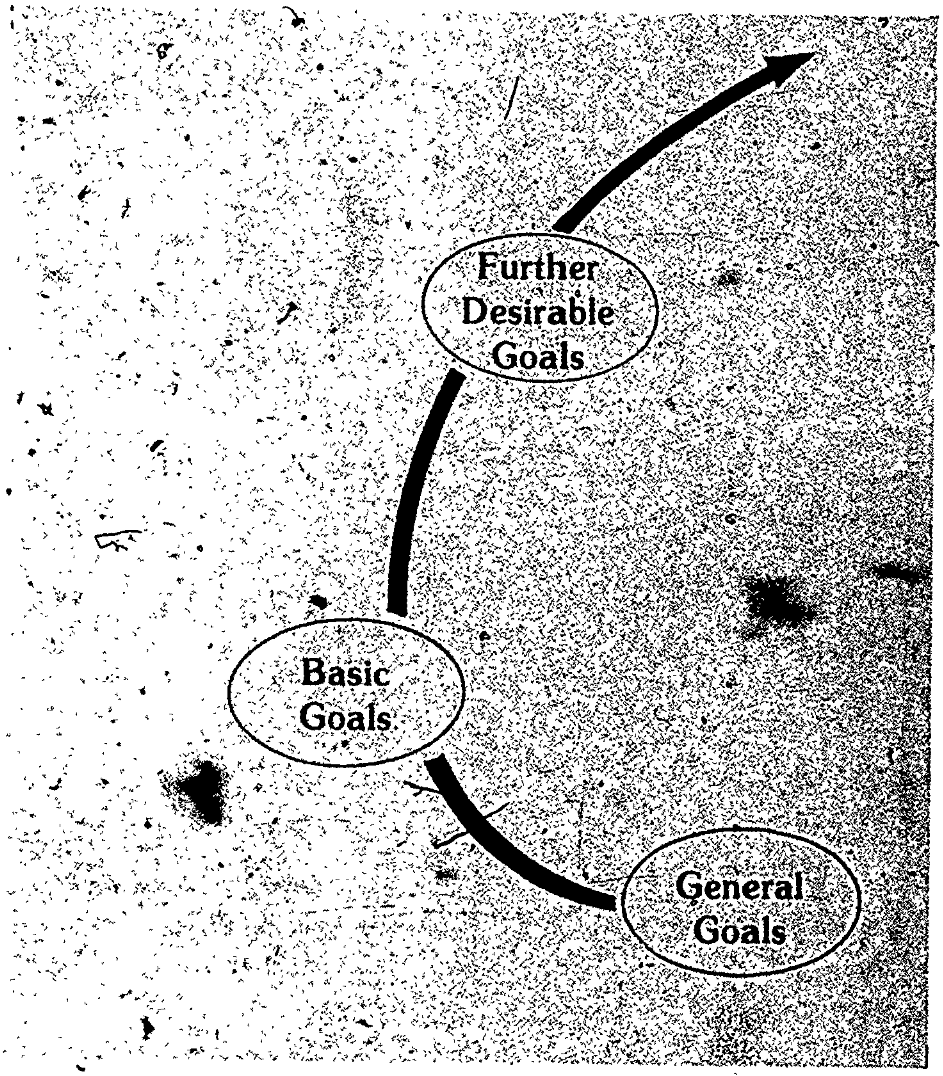
Task	Responsibility of	Resources in Mathematics for Georgia Secondary Schools
1. Formulate mathematics curriculum committee composed of • local curriculum director • local mathematics supervisor or designated mathematics curriculum leader • representatives from mathematics teachers in elementary, middle and secondary schools • media specialist • representatives from other curriculum areas (to be called on as needs arise) • guidance counselor (to be involved periodically).	Curriculum leaders, general and mathematics	
2. Develop goals of mathematics learning. • Study general and mathematics goals of local and state educational agencies and state and national professional mathematics organizations. • Study (or formulate) philosophy of school system regarding general education and mathematics education. • Consider local student present and future needs.	Mathematics curriculum committee	Goals, Problem Solving, Points of View, Resources, Appendices
3. Review state and local high school graduation requirements and statewide Criterion-Referenced Test objectives.	Mathematics curriculum committee	
4. Study materials and provide sufficient time to discuss findings regarding such questions as the following. • How do students learn mathematics? How do students learn mathematics as a language? As a science? As a collection of skills? As an art? • What are students' present attitudes toward mathematics? What changes in attitudes and appreciations do you wish a modified mathematics program to attain? • Are there specific mathematical needs for your community? Are there particular needs in careers typically pursued by your students? • Do all present courses include sufficient opportunities for problem solving and evaluation of problem solving?	Subcommittees of mathematics curriculum committee	Problem Solving, Strategies, Evaluating, Points of View, Objectives, References, Appendices

• What strategies of teaching should be employed? Are a variety of strategies used in teaching each course? • What major topics of mathematics should be addressed in the curriculum? Where should these be addressed? **Note: Keep notes on readings to help in writing courses, especially activities and references.** These findings should provide a framework within which the curriculum can be built.		
5. Develop student objectives for mathematics education and indicate those essential skills expected of graduating seniors, whether they enter the world of work or postsecondary schools.	Mathematics curriculum committee and consultants	Goals, Problem Solving, Points of View, Objectives, Appendices, Resources
6. Review existing curriculum to find out if essential skills are included in appropriate courses. Indicate those missing from curriculum.	Members of mathematics curriculum committee	
7. Review existing curriculum in terms of stated goals, objectives and local student needs. Indicate inconsistencies.	Members of mathematics curriculum committee	
8. List courses to fulfill needs of local students. Some may be mini-courses linked together for one quarter, one semester or one year, according to school size and organization. Examples: general mathematics (a collection of mini-courses); laboratory mathematics; personal finance; courses built around applications — mathematical modeling, consumer mathematics, mathematics of traveling to foreign countries; mathematics used in sports; series of algebra courses; geometry courses appropriate for local students; trigonometry as mini-course (one course might be for those entering technical school); computer literacy and other computer courses; history of mathematics; probability and statistics (various levels); language and mathematics; introduction to logic; number theory, combinatorics; interdisciplinary mathematics and independent study	Subcommittees and the committee of the whole of the mathematics curriculum committee	Strategies, Points of View, Sample Courses, Appendices
9. Write tentative course plans using information and writings from previous steps. Plans should include • title — succinctly reflecting nature of course; • course description — brief narrative to be included in course catalog;	Subcommittee of mathematics curriculum committee	Problem Solving, Strategies, Evaluating, Points of View, Objectives, Sample Courses, Resources, Appendices, Careers in Mathematics

- course objectives — each containing condition, task, level of acceptability and each keyed to student competency(ies) required for graduation;
- course content — outline of topics;
- instructional activities — relating to objectives;
- procedures for evaluating courses — representing methods of assessing student achievement of objectives;
- resources — including print and nonprint media, equipment and human resources helpful in achieving student objectives.

10. Review tentative course offerings and respond to the following questions. • Have appropriate offerings been provided for all levels of students? • Can appropriate courses be scheduled each quarter or semester for all students? • Have courses been identified which match minimum requirements for graduation? Do these courses include the competencies required for graduation? • Are courses planned to allow for as much flexibility in scheduling as possible? • Are course objectives stated so that evaluation of student attainment can be measured? • Do the courses provide opportunities for a variety of strategies including discovery approach, small group or individual activities, observation, exploration, investigation, inquiry, organization of ideas, organization of data, applications to other disciplines, reinforcement? • How can the level of student involvement be increased? • Are student activities appropriate with respect to needs, abilities and interests? • Based on present inventory are all needed materials on hand? If not, list missing materials and rank them from most to least needed.	Subcommittees and the committee of the whole of the mathematics curriculum committee, curriculum committee, additional mathematics teachers and consultant(s)	Problem Solving, Strategies, Evaluating, Points of View, Objectives, Course of Study, Resources, Appendices, Careers in Mathematics
11. Revise the tentative courses on the basis of responses to task 10 above.	Subcommittees of mathematics curriculum committee	
12. Develop a plan to fieldtest the program.	Subcommittee of mathematics curriculum committee	
13. Select schools and teachers to field-test the program.	Administrators, mathematics curriculum leader	

14. Field-test the program; keep notes regarding changes needed in the program.	Designated personnel	
15. Review/revise curriculum; use questions in previous steps to develop plan for review/revision.	Mathematics curriculum committee as a whole	
16. Plan for evaluation of mathematics curriculum.	Subcommittee	Evaluating, Objectives
17. Formulate and implement staff development plan.	Subcommittee of mathematics curriculum committee, mathematics curriculum leader, appropriate administrators and all of mathematics staff	
18. Implement the mathematics curriculum plan.	Appropriate administrators and all of mathematics staff	
19. Evaluate the mathematics curriculum each year.	Appropriate administrators, mathematics curriculum leader and designated staff members.	Evaluating
20. Review findings of evaluation each year and plan revision where needed.	Designated personnel	


Further
Desirable
Goals
Basic
Goals
General
Goals

Goals of Mathematics Learning

In 1979 Shirley Hill, president of the National Council of Teachers of Mathematics, presented a challenging essay on how teachers should react, as professionals, to forces attempting to narrow the mathematics curriculum through demands for higher test scores and accountability. She pointed out that the quality of education could decline even as scores on minimal competency exams improved if, for example, these gains in scores were at the expense of long-term retention and application. With regard to the second National Assessment of Education Progress she stated, "Results remain good in whole numbers computation, but performance in applying these skills to multistep, nonroutine or even fairly routine problem solving was dismal." She concluded that the public in general and parents in particular must be alerted to the danger of developing a generation of test passers who cannot use or apply assiduously drilled low-level skills. What is the role of professionals in combating this possibility? Hill stated, "First, we must not compromise our goals and objectives for mathematics learning. We have a good idea of the abilities that our students will need a decade or so hence; we can predict that much of what has been basic will become obsolete. We know the necessity for problem-solving and application abilities and the limitation of a curriculum dominated by computational skills. We know the hazards of concentrating on low-level skills and minimums."

Hill's statement raises a crucial point for those engaged in design of mathematics curriculum. That is, what should be the goals of mathematics learning? One possible response is provided in the following paragraphs.

A conference on basic mathematical skills and learning held in 1975 in Euclid, Ohio, and known as the Euclid Conference, addressed this major concern of identifying goals. The results of this conference have had a significant influence on the mathematics education community. The conference participants identified three goal categories.

General goals indicate those advantages that grow with an increasing understanding of mathematics.

Basic goals are the mathematics needed by workers, consumers and citizens.

Further desirable goals are those that will meet the needs of students whose vocational requirements and interests go beyond the basic goals.

The following sections will clarify the intent of the three goal areas.

General Goals

The purpose of teaching mathematics is to prepare students for living as consumers and citizens, inaugurate education for various productive occupations and aid students in developing worthy and satisfying lives. The purpose of mathematics education is to provide experiences which enhance students' perceptions, aid them in constructive reasoning and bring insight to a variety of situations and problems. Additionally, experiences in mathematics should be such that students feel confident in situations where reasoning and quantitative thinking are needed. Students should develop a level of self-confidence needed to function effectively in a society in which heavy use is made of mathematics and mathematical concepts. Mathematics programs must include those mathematical tools that are useful and are needed to cope with realistic problems.

Basic Goals

The Euclid Conference report (Euclid Conf., 1975., pp. 7-20) identified 10 basic goals.

1. **Appropriate computational skills.** "The automation of arithmetic during the past half century has strongly affected educational needs" (Euclid Conf., 1975, p. 17). The advent of the hand-held calculator has had enormous effects on society and is a complex issue for mathematics education. The Euclid Conference participants charged mathematics educators to find the best combination of skills and understandings so that a student might be able to develop a needed algorithm and make use of time-saving devices such as hand-held calculators.

2. **Links between mathematical ideas and real-world situations.** Students should be able to relate mathematical properties to real-world situations. This involves expressing a situation in terms of mathematics as well as manipulating the mathematics to arrive at a conclusion. To gain some insight into the original situation students must be able to translate conclusions (or results or solutions) into the terms of the original situations.

3. **Estimation and approximation.** These skills are basic to dealing with quantitative ideas. Students must carry out rapid, approximate calculations and acquire awarenesses of the notions of error and precision.

4. **Organization and interpretation of numerical data, including using graphs.** Students need to be able to use numerical data to set up charts, tables and graphs. They must be able to read such a display and draw pertinent conclusions.

5. **Measurement, including selection of relevant attributes, selection of degree of precision, selection of appropriate instrument, techniques of using measuring instruments and techniques of conversion among units within a system.** Often, mathematicians and mathematics educators consider measurement a legitimate topic of science. However, the importance of measurement is not in question. Students should be able to measure length, distance, weight, volume, temperature, time, area and perhaps angles as well.

6. **Alertness to reasonableness of results.** Students must learn to inspect results, conclusions and solutions for reasonableness of answers in terms of the original situation.

7. **Qualitative understanding of and drawing inferences from functions and rates of change.** The concept of how one quantity may depend on another is necessary to understand and interpret situations. Understanding the relationships among quantities as represented in terms of tables and graphs is necessary.

8. **Notions of probability.** Forecasts and predictions are based on the notions of probability which frequently arise in problem-solving situations.

9. **Computer uses: Capabilities and limitations (gained through direct experiences).** The impact of the development of the computer affects all aspects of our society. All citizens must understand what computers can and cannot do and, in particular, that the performance of computers is governed by people.

10. **Problem Solving.** Problem solving is the unifying goal interrelating not only these 10 basic goals but also interrelating the general, basic and further desirable goals.

Further Desirable Goals

The basic goals presented in the preceding section stress the relationship between mathematics and the real world. It is also desirable that students know about the discipline of mathematics and its internal considerations. The Euclid Conference report (Euclid Conf., 1975, pp. 20-21) cites the following as five further desirable goals.

1. **Recognition that mathematics is a construct.** Mathematics is a product of inquiring minds and a reflection of the progress of humanity. Students should understand that while mathematicians determine assumptions upon which mathematics is based, this determination is not capricious and is in fact an attempt to develop mathematical structure which are internally consistent.

2. **Ability to reason abstractly.** Students should understand the nature of argument (or proof) and should be able to make judgments about the reasonableness of various arguments. Fields other than mathematics make use of the ability to construct such logical arguments.

3. **Enrichment of the students' world.** Students' lives are enriched by knowledge of the contributions to culture that mathematicians have made through the mathematics that they have developed.

> Mathematics can be an aid to insight — a way of looking at events and phenomena that brings increased appreciation, understanding and creativity. Developing such styles of perception is, or should be, part of what it means to become educated (Euclid Conf., 1975, p. 21).

4. **Acquaintance with the natural notations of mathematics.** As mathematicians have developed mathematics over the centuries they have arrived at generally accepted ways of communicating mathematical ideas. Examples of symbols and rules relating the use of these symbols are the Hindu-Arabic notation, including use of the numeral for zero and the use of exponents. These notations have greatly aided the communication of mathematical thought.

5. **Mathematical modeling.** A mathematical model is an approximate translation of a real-world situation into mathematical terms. It is through this modeling process that mathematics is most elegantly and usefully applied to the changing needs of the world.

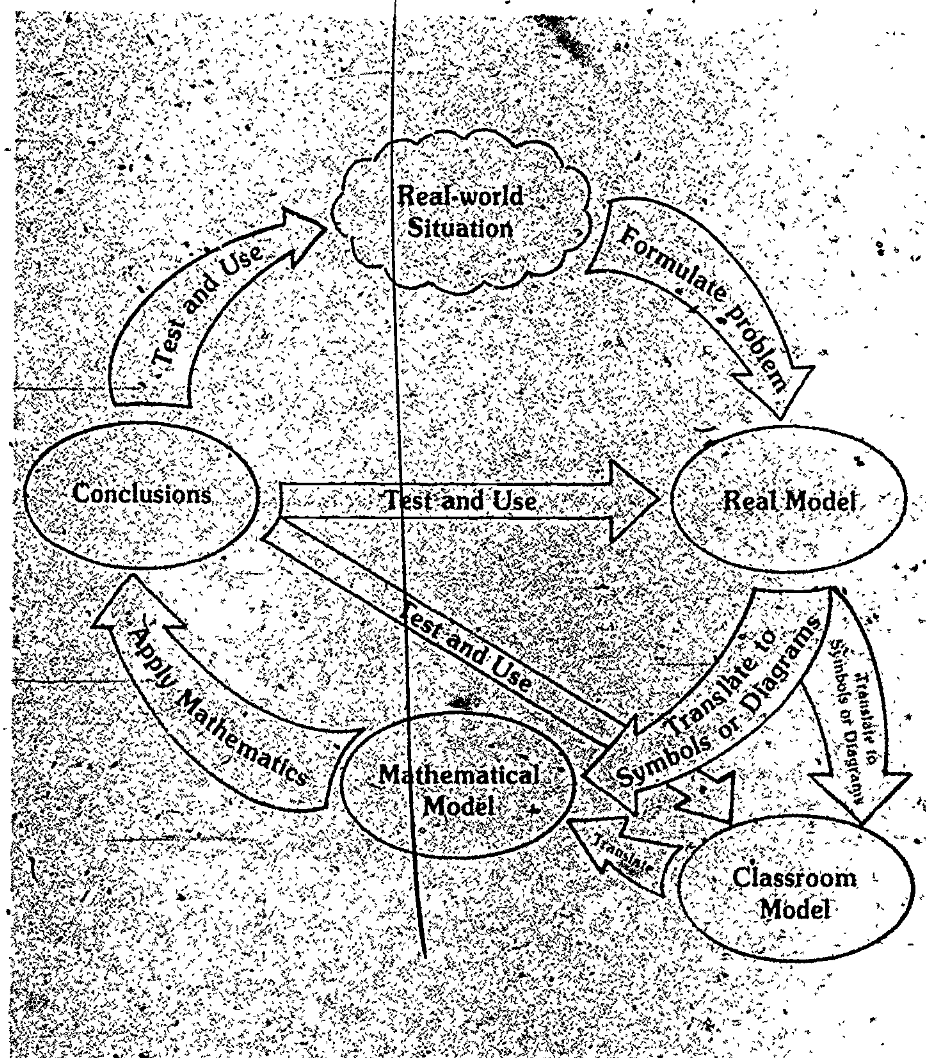
A prime example of the influence of the above goals is that both the National Council of Supervisors of Mathematics and the National Council of Teachers of Mathematics endorsed, in a one-to-one fashion, the Euclid Conference's second category of 10 basic skill areas. With regard to problem solving, which these groups placed first on their list of basic goals, the following strongly asserted statement was made.

> Learning to solve problems is the principal reason for studying mathematics. Problem solving ~~is the process of~~ applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with nontextbook problems. Problem-solving strategies involving posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny. (NCTM and NCSM position papers, "Basic Skills", *The Arithmetic Teacher*, Volume 25, No. 1, Oct. 1977, p. 20).

Even a cursory examination of the above goal statements shows that there is clear call for engaging students in problem solving and applications.

For example, to realize the two basic goals of problem solving and links between mathematical ideas and real-world situations the remaining eight basic goals all must play their part. For certainly such processes as estimating, approximating, graphing, measuring, drawing inferences and assessing reasonable results all come into play in applying mathematics.

The next section specifies a theme for mathematics instruction using the ideas and goals of this section as a rationale for its selection.


Real-world Situation
Formulate problem
Real Model
Translate to Symbols or Diagrams
Translate to Symbols or Diagrams
Classroom Model
Translate
Mathematical Model
Apply Mathematics
Conclusions
Test and Use
Test and Use
Test and Use

Stressing Problem Solving and Applications

This section of the guide presents a theme which may provide mathematics teachers with their most challenging and exciting opportunity to make teaching more effective and to improve curriculum. This theme centers around teachers trying to make mathematics increasingly useful to those who learn it. How can this be done? The claim is that if teachers make greater provision for stressing problem solving and applications in their teaching approaches, then students will have a greater likelihood of obtaining a worthwhile awareness and appreciation for both the spirit and uses of mathematics. In particular, this section attempts to show that there exists a variety of situations and problems that have the potential to engage students' interests. Also, this section will try to show that much of the fairly standard material currently in secondary programs provides a tool kit which — when properly guided by an enthusiastic and knowledgeable teacher — can powerfully aid students in tackling real problem situations of concern to the student or future adult.

What is problem solving?

Research on problem solving rests on the assumption that most mathematical activity is problem solving, that effective teaching of mathematics is associated with having students develop and use problem solving techniques.

Based on seminal ideas of Karl Duncker (1945), John F. Lucas (1972) stated that, in general, a problem situation exists when one possesses certain given information and a goal but lacks a connection between the two. A solution of the problem results when an individual establishes a meaningful connection between information and goal. The term *problem solving* implies more than seeking a solution. In fact, the process of solving a problem involves an active search for a suitable method for dealing with the problem — and subsequent application of that method. Prior experience may provide appropriate methods, procedures and plans. However, for many mathematical problems an individual must construct and test a variety of plans before one is found to be adequate for the problem situation at hand. Consideration of such plans leads many researchers to the notion of heuristics or maxims — which are examined next.

Problem-Solving Maxims

One of the most thoughtful and thorough treatments concerning the teaching of problem solving has been developed by T. J. Cooney, E. J. Davis and K. B. Henderson in their text *Dynamics of Teaching Secondary School Mathematics* (Houghton Mifflin Company, Boston, 1975, pp. 240-291). These authors present *maxims* which students can use in their problem solving activities. They believe, along with George Polya, that if students become aware of the maxims and learn to incorporate the maxims into their problem-solving behavior then students will become more effective problem solvers. The teacher should serve as a model by using the following maxims when teaching students to solve problems.

Make sure students understand the problem.

- Do the students understand the meaning of terms used in the problem? What ambiguous terms need clarifying or illustrating?
- Do the students identify all relevant given conditions and information?
- Do the students determine what they are being asked to find? Would a solution to the problem take the form of a number, a set of numbers, an equation, a graph, a pattern or what mathematical object? How can such a solution be generated?
- Can students restate the problem in their own words? Would a sketch be helpful in explaining the problem? Cooney, Davis and Henderson conclude, "If students can indicate they know the meaning of all the terms stated in a problem and can identify the given information and the nature of the required solution and additionally can express the problem in their own words, then the teacher has a substantial basis for assuming that the problem is understood" (Cooney, et al, 1975, p. 248).

Help students gather relevant ideas to assist in creating a plan.

- What information can be derived from the given conditions? What relations or implications can be deduced from the given conditions or from the assumed type of solution? Should more brain storming take place? What subset of generated information is most likely to be relevant? Are students being encouraged to generate novel plans for attacking the problem? Can more than one plan be devised? Has there been an attempt to form a plan by deriving information from joint consideration of the given and the sought after solution to the problem? Are slower students being brought into the planning stage? Is the teacher telling too much? Would a different or more elaborate sketch be helpful?
- Can students gain information by solving an analogous problem? Can students invert from a problem involving an ellipse to one involving a circle and then — after gathering some ideas and cues to solution — invert back to the ellipse problem?

  Is it possible to try smaller values of some variable in order to see a pattern? (this is one of the most powerful techniques in problem solving!) Can the pattern be verified?
- Should students attempt a different approach after they become blocked or discouraged? Are they in a mind set which needs breaking? What would be a different perspective for considering a particular problem? Is an approach being abandoned too soon? (Most challenging problems required persistence as well as insight.)

Provide an atmosphere conducive for students to solve problems.

- Are the students receiving positive cues from the teacher? Are they being encouraged to continue their approaches? Are teachers squelching approaches too much?

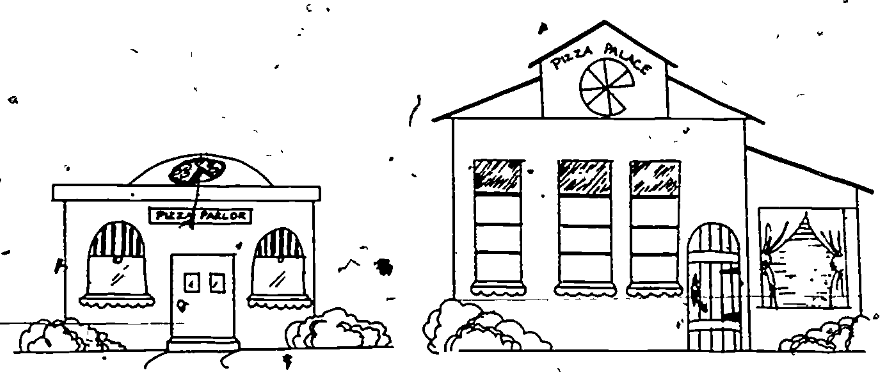
- Are students being given sufficient time to deal with a problem? Should more time be provided for ideas to incubate?
- Are students being taught guessing? (Polya has stressed this point as being crucial.) Is the teacher encouraging and rewarding students who make conjectures? Is the teacher helping students to test their conjectures? Are teachers being honest with regard to their own false starts? Is there enough emphasis on the inductive made?
- Are students being made aware that there are usually many ways to solve a problem? Are they being encouraged to seek alternate solutions? Are students or the teacher solving the problems? Is there adequate discussion of the processes involved in the solution stage?

After a solution is obtained encourage them to look back and reflect on the problem and the means used to solve the problem.

- Does the solution make sense? What other means could be used to verify the solution?
- Does the conclusion obtained really solve the problem? Should the problem be reformulated and the problem-solving cycle be repeated?
- What insights about the original problem and related problems are provided by the solution?
- What are other ways to solve the problem? What occurs if you focus on some other feature of the problem? Does that suggest a different approach?
- If the problem was solved inductively can a deductive solution be determined (and vice versa)?
- What subproblems were suggested and dealt with in obtaining the solution?
- What new problems can now be generated? What if some aspect of the problem were changed?
- How can the problem and solution be applied and to what? What occupations might encounter such a problem?
- What would be different if a calculator or computer was available?
- Have there been sufficient classroom discussions about the critical role of employing maxims in dealing with problems?
- What questions or remarks from students could be exploited by the teacher to help students seek solutions or formulate new problems?
- Were the students having fun in their problem-solving activity?
- Are students aware that the maxims used in the mathematic classroom apply to almost any challenging life situation they may encounter?

The next section provides a very complete treatment of just one problem and illustrates how many of the maxims identified above can be applied.

Problem Solving in a General Mathematics Classroom



Situation

To see that problem-solving maxims can be used at all levels of instruction, consider the following situation which involved the purchase of pizza by students in a general mathematics classroom. They determined, after a rather heated debate, that both the Pizza Palace and the Pizza Parlor restaurants had the best tasting pizzas. The following problem was next examined.

Problem

Did the Pizza Palace or the Pizza Parlor offer the best buy on pizza?

Understanding the Problem

It was pointed out that the word *best* in the above problem had to be clarified. In a brainstorming session many considerations were identified and discussed by the students. Among the considerations listed were the following. (1) price, (2) size of pizza (i.e., diameter measured in inches), (3) number of slices for a given size of pizza, (4) thickness, (5) cost for various types of topping (e.g. pepperoni, mushroom, Italian sausage, onion, green pepper, anchovy, shrimp), (6) number of toppings, (7) distance to restaurants and (8) carry-out charge. But which of these should the class focus on?

It was decided that to determine the best buy between the two restaurants data must be gathered to find out the cost per square inch for different sizes of pizza and different number of toppings. Also, toppings which cost extra, such as shrimp, were to be ignored.

Creating a Plan

To begin to determine cost per square inch it was decided to gather prices for a small pizza, a medium pizza and a large pizza at both restaurants. The medium and large pizzas at both restaurants were, respectively, 13 inches and 15 inches in diameter. However the small at the Pizza Parlor had a nine-inch diameter whereas the small at the Pizza Palace had a 10-inch diameter.

Since the area A of a circle is πr^2 and $r = d/2$, students saw that the formula

$$A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$$

could be used to determine the total area of a pizza for a given diameter. Then the various pizzas could be compared by computing Cost/Area for each given diameter. In addition to size, students knew that the cost depended on the number of toppings requested. Letting t denote the number of toppings students decided to find costs for values of t ranging from t = 0 through t = 4.

Thus the plan for determining which restaurant had the best buy on pizza called for comparing each restaurant's cost per square inch of pizza with respect to size and number of toppings.

Carrying Out the Plan

Students brought the following information back to class.

Pizza Parlor

No. of Toppings	**Size 9 inch**	**13 inch**	**15 inch**
0	\$3.25	\$5.00	\$ 6.40
1	\$3.75	\$5.70	\$ 7.30
2	\$4.25	\$6.40	\$ 8.20
3	\$4.75	\$7.10	\$ 9.10
4	\$5.25	\$7.80	\$10.00

Pizza Palace

No. of Toppings	**Size 10 inch**	**13 inch**	**15 inch**
0	\$2.95	\$4.30	\$5.65
1	\$3.90	\$5.25	\$6.70
2	\$4.45	\$5.85	\$7.30
3	\$4.85	\$6.40	\$7.95
4	\$5.25	\$6.65	\$8.30

To compare the two restaurants with respect to 13-inch pizzas with four toppings, one calculates that a 13-inch pizza has an approximate area of 133 square inches.

$$A = \frac{\pi d^2}{4} = \frac{\pi(13)^2}{4} = 133.$$

Then for the Pizza Parlor the cost per square inch of a 13-inch pizza with four toppings is \$7.80/133 in^2 = \$ .06/in^2

Using a hand-held calculator students found the following cost per square inch for the two restaurants.

Pizza Parlor

No. of toppings	9 inch (64 sq. in.)	13 inch (133 sq. in.)	15 inch (177 sq. in.)
	Diameter (area)		
0	$ .05	$ .04	$ .04
1	$ .06	$ .04	$ .04
2	$ .07	$ .05	$ .05
3	$ .07	$ .05	$ .05
4	$ .08	$ .06	$ .06

Pizza Palace

No. of toppings	10 inch (79 sq. in.)	13 inch (133 sq. in.)	15 inch (177 sq. in.)
	Diameter (area)		
0	$ .04	$ .03	$ .03
1	$ .05	$ .04	$ .04
2	$ .06	$ .04	$ .04
3	$ .06	$ .05	$ .04
4	$ .07	$ .05	$ .05

Examination of the above two tables showed that the Pizza Palace appeared to offer the better buy since in each case the cost per square inch to the buyer was less than or equal to the corresponding figure at the Pizza Parlor. However, the costs per square inch values were found to be misleading. For example, the cost for a 15-inch pizza with one topping was estimated at four cents per square inch — at both restaurants — even though the parlor charged $ .60 more than the Palace for this particular pizza.

The overall conclusion was that the approach used, i.e., comparing costs per square inch, provided some interesting information but did not give a precise answer to which pizza was the best buy. Also, for pizzas of the same size and number of toppings one need only use the price list to compare the two restaurants. Students did note that the worst buy was the Pizza Parlor's nine-inch pizza with four toppings. This pizza costs eight cents per square inch — the highest such cost. Overall, the students became more aware of comparison shopping and had enjoyed working with the problem.

Looking Back

Among the suggestions made why the cost per square inch approach was not effective were the following. (1) the assumption that pizzas are circles, (2) errors made in approximating diameters and areas and (3) round-off errors.

One person thought that thickness should have been used. Then the model for a pizza would be a cylinder and one would estimate cost per cubic inch.

Many students noted that the Pizza Parlor's price list had many regularities. For example, each additional topping for the nine-inch pizza cost 50 cents. Also for the 13-inch and 15-inch pizzas additional toppings cost, respectively, 70 cents and 90 cents. (No such pattern was found for the Pizza Palace.)

When the Pizza Parlor's prices were given to an algebra class they found if they doubled the price of a nine-inch pizza, denoted by P_9, and then subtracted multiples of 10 cents they could predict the price of the 15 inch pizza (denoted by P_{15}). The formula they obtained was

$$P_{15} = 2 P_9 - .10 (t + 1) \text{ where } t = \text{no. of toppings.}$$

Also they found the price of 13-inch pizza (P_{13}) depended on P_9 and t. In fact

$$P_{13} = 2 P_9 - .30 (t + 5).$$

One group examined P_{15} and P_{13} above and then conjectured if the Parlor sold a 14-inch pizza its price would be

$$P_{14} = 2 P_9 - .20 (t + 4).$$

They obtained this equation for P_{14} by adding P_{13} to one half the difference of P_{15} and P_{13}. Note the value of P_9 used in the above formulas depends on the value of t being used. For example, to find the cost of a 14-inch pizza with three toppings, substitute the price of a nine-inch pizza for t = 3. Then

$$\begin{aligned} P_{14} &= 2 P_9 - .20 (t + 4) \\ &= 2 (4.75) - .20 (7) \\ &= 9.50 - 1.40 = \$8.10. \end{aligned}$$

One nice feature of this conjectured formula for P_{14} was that each topping cost a fixed price of 80 cents.

The teacher asked what graphs could be constructed to help determine the price of a giant 17-inch pizza. Many different linear graphs were presented. One nonlinear graph appeared to indicate that a zero-inch pizza would, surprisingly, have a positive cost. This led to a discussion of fixed costs. In summary, questions involving the term *best* offered a fruitful area for involving students with real-world problem-solving experiences.

An Analysis of a Rate of Change Problem

Situation

The following details how one person approached and solved a rate of change problem. Because many teachers do not regularly teach this topic it is hoped that the problem presented will encourage them to do so.

Before reading the solution one should attempt to solve the problem. While making this attempt the reader is urged to keep track of those strategies which occur at various stages in the process of solving the problem. Of course, a variety of strategies are possible for any problem. But it may prove instructive to compare your approach with the heuristics and strategies identified in the discussion which follows.

Problem

The legs of a right triangle are initially five inches and 12 inches long. If the short leg is increasing uniformly at the rate of one inch per second and the long leg is decreasing at the rate of two inches per second, how fast is the area changing when the triangle is isosceles?

Understanding the Problem

What is given? A right triangle whose legs are initially five inches and 12 inches long. The shorter leg is increasing at the rate of one inch per second and the longer leg is decreasing at the rate of two inches per second.

Draw a diagram; introduce suitable notation. The lengths of the legs are varying, so let y represent the length of the shorter leg and let x represent the length of the longer leg.

Initially, $y = 5$ and $x = 12$.

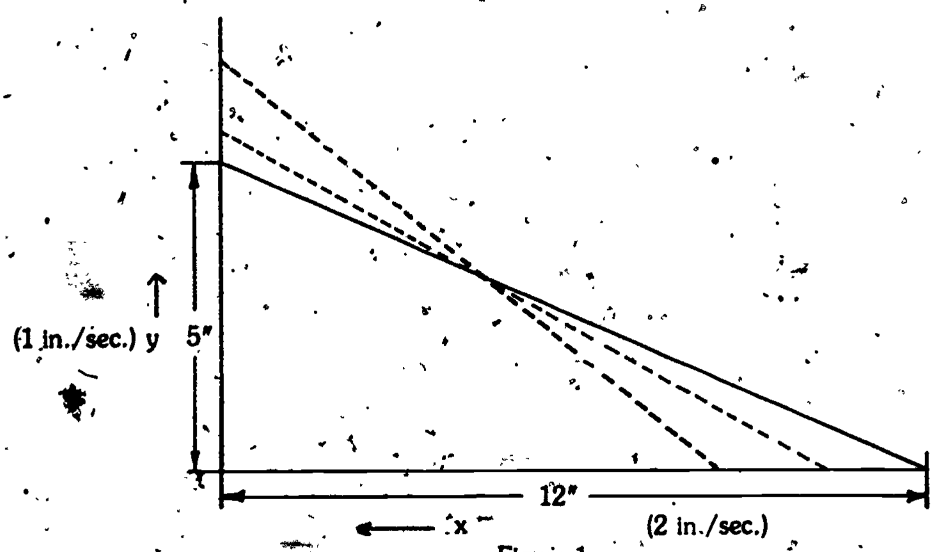


Figure 1.

Since the rates of change are specified in length per unit of time, we are also given,

$$D_ty = 1 \text{ inch per second,}$$

$$D_tx = -2 \text{ inches per second.}$$

Creating a Plan

What is required? The rate of change of the area of the triangle at the moment when the triangle is isosceles. Since area is changing with respect to time, let A represent the area of the triangle. We want, D_tA when $x = y$, and where D_tA denotes the derivative of area A with respect to time t. One way to begin devising a plan is by *reasoning analytically*. Problem solving involves making a connection between what is given and what is required. Suppose we already have the rated D_tA, D_ty and D_tx related in some way, perhaps by an equation. Can we find some equation which may have led to this relationship? Evidently, we need some connection between A, x and y so that we could differentiate with respect to time and produce the relationship involving derivatives. (This is the method often tried for problems involving rates of change.) Is there any relationship between A, x and y which can be seen? We are given a right triangle and we want to know something about its area. Since y is the length of its height, x is the length of its base and the area of any triangle is one half the height times the base (recall this result from prior experience and use it now), this is the relationship we want

$$A = \frac{1}{2} xy.$$

Carrying out the First Plan

Differentiating with respect to t (area, side x and side y are all changing with respect to time, hence are functions of t, and it makes sense to differentiate with respect to t, we have

$$D_tA = \frac{1}{2} [xD_ty = yD_tx].$$

Now we have produced the connection planned for earlier; that is, we have connected the known rates D_ty and D_tx with the unknown rate D_tA. Are we done? Can we directly find out what D_tA is? The terms x and y are also involved in the equation. What can we do with them? Perhaps we should check back to see if we have used all the available information. Checking, we find that we have not used the original values of x and y, nor have we used the fact that we want D_tA when the triangle is isosceles, i.e., when $x = y$.

So, it appears that we are confronted with a new problem. We have D_tA related to D_ty and D_tx (D_tA is the general derivative of area with respect to time) we want a specific value of D_tA when $x = y$. If we knew the specific values of x and y when the triangle is isosceles, we could immediately find D_tA at that time by making replacements in the equation we already have (analytic reasoning). How can we obtain these values? Well, we cannot just replace x by 12 and y by 5 (the initial values) because the triangle is not isosceles in the beginning. What is really happening here? Since one side of the triangle is increasing and the other decreasing, we can think of the triangle as having sides that act like elastic or springs, where someone is stretching the short side at a certain rate (one inch per second) and compressing the long side at a certain rate (two inches per second). At some particular instant the sides are equal. Before that x is longer and y is shorter; after that instant x is shorter and y is longer. If we can find out at what time or t (in seconds) the sides are equal, we might be able to find the lengths at that instant, and thereby solve the problem.

Carrying out the Second Plan

To carry out this plan, we might try some approximations. Initially, when $t = 0$ seconds, $x = 12$ inches and $y = 5$ inches. When $t = 1$ second, $x = 10$ inches and $y = 6$ inches, since x is decreasing at a rate of two inches per second, and y is increasing at a rate of one inch per second. We would continue this, but perhaps we can generalize. After t seconds, the vertical side is of length $y = 6 + 1t$ (the 1t is a rate times time, hence a distance); and the other side acts similarly (only decreases) so it is of length $x = 12 - 2t$. Now, we wanted to find the specific value of t when the sides are equal, so the problem reduces to that of finding t under the condition,

$$5 + t = 12 - 2t,$$

from this $$3t = 7;$$

therefore, $$t = \frac{7}{3} \text{ seconds.}$$

Finally, we can now carry out the total plan. Since we know t when the triangle is isosceles, we can find x and y at this instant. We have,

$$y = 5 + t = 5 + \frac{7}{3} = \frac{22}{3} \text{ inches,}$$

and $$x = 12 - 2t = 12 - 2\left(\frac{7}{3}\right) = \frac{22}{3} \text{ inches.}$$

Since we have $$D_tA = \frac{1}{2}[xD_ty + yD_tx],$$

and we were given $$D_ty = 1,\ D_tx = -2,$$

and we found $$x = y = \frac{22}{3} \text{ (when the triangle is isosceles),}$$

then
$$\begin{aligned} D_tA &= \frac{1}{2}\left[\left(\frac{22}{3}\right)(1) + \left(\frac{22}{3}\right)(-2)\right] \\ &= \frac{1}{2}\left[-\frac{22}{3}\right] \\ &= -\frac{11}{3} \text{ square inches per second.} \end{aligned}$$

Looking Back

There are many ways of checking this result. First, is this result reasonable? The $-11/3$ implies that the area is decreasing with respect to time. We can check several instances to see if this is true.

At $t = 0$, $A = 30$ square inches.

At $t = 1$, $A = 30$ square inches.

At $t = 2$, $A = 28$ square inches.

At $t = \frac{7}{3}$, $A = 26.9$ square inches.

At $t = 3$, $A = 24$ square inches.

So, at $t = 7/3$, the area is decreasing with respect to time, and the rate of decrease does appear to be about four square inches per second.

Next, did we use all the information? Yes, the given rates D_tx and D_ty were used to obtain the relationship,

$$D_tA = \frac{1}{2}\,[xD_ty + yD_tx].$$

With the initial values $y = 5$ and $x = 12$, together with the condition that the triangle be isosceles at some instant, we generated a method to find the specific values of x and y at that instant. Try a test of dimensions. In the expression,

$$\frac{1}{2}\,[xD_ty + yD_tx],$$

½ is dimensionless, x is in inches, and D_ty is in inches per second. So we obtain square inches per second for the dimensions of xD_ty. By symmetry yD_tx is also in square inches per second, so that the result D_tA should be in square inches per second — and it is.

In addition to checking the result, we can apply various techniques to check the equations we have used to see if they really represent the situation described in the problem. One method for doing this is investigating any symmetry in the data of the problem. Since x and y correspond to the lengths of sides of a right triangle, either could have been used to represent the height or the length of the base. So, the roles of x and y are interchangeable. Is this symmetry (interchangeability) reflected in the equations? Yes, since in both

$$A = \frac{1}{2}\,xy$$

and

$$D_tA = \frac{1}{2}\,[xD_ty + yD_tx]$$

the roles of x and y are symmetric.

Another check of the equations can be done by investigating special cases. Suppose we hold one side fixed and let the other change. For example, if x were constant at 12 inches, and y were increasing at the rate of one inch per second — we would expect the area to increase, and

$D_tA = \frac{1}{2}\,[0 + yD_ty + 0] = \frac{1}{2}\,[12\,(1)] = 6$ square inches/second supports this expectation.

On the other hand, if y were fixed at 5 inches and x were decreasing at the rate of 2 inches per second, then since y is constant, $D_ty = 0$, and

$D_tA = \frac{1}{2}\,[0 + yD_tx] = \frac{1}{2}\,[5(-2)] = -5$ square inches / second.

Since the original area of the triangle was 30 square inches, it should take six seconds for the triangle to collapse — and it does, since x is originally 12 inches long and is decreasing at the rate of two inches per second (the value of x is zero in six seconds).

In addition to checking the result or steps of the solution, there are other ways of looking back at the problem. We can vary the solution process by condensing it (to see it at a glance) or by attempting to solve the problem in another way. For example, a condensation might go as follows. Given a right triangle with legs, to find the rate of change of its area (under special conditions),

(1) use the area formula.
(2) differentiate with respect to time to produce a relationship between derivatives.
(3) find the particular time-value for which the special conditions hold.
(4) use the values of time to determine the lengths of legs under the special conditions.
(5) evaluate the derivative of area with respect to time.

In the problem just solved, area was a function of three variables, namely, t, x and y. Can we solve the problem in another way? Perhaps we can express A as a function of t alone. Since both legs are changing with respect to time, let t represent time in seconds. Then, at any time t, the side lengths are $5 + t$ and $12 - 2t$ (using the given rates, -1 and 2). In particular, when the triangle is isosceles, we have,

$$5 + t = 12 - 2t$$

$$t = \frac{7}{3} \text{ seconds.}$$

Now,

$$A = \frac{1}{2}(5 + t)(12 - 2t)$$

$$= (60 + 2t - 2t^2)$$

$$= 30 + t - t^2.$$

The last equation expresses A as a function of the single variable t. Therefore,

$$D_tA = 1 - 2t.$$

And when $t = 7/3$, $D_tA = 1 - 2[7/3] = -11/3$ square inches per second. (We might note that this alternate solution did not require implicit differentiation.)

Finally, we can look back at the problem to extend to analogous problems that which we learned; that is, we can try to vary the problem, inventing a new problem analogous to this one. For example, we could take the case where both sides are increasing and try to find the time when the longer side is four times as long as the shorter. Another variation arises by dropping the condition that the triangle be a right triangle; or we could invent a new problem by analogy to a physical situation. The following problem is an example. Suppose one plane is 500 miles north of Chicago and traveling northward at a uniform rate of 100 miles per hour Another plane is 1200 miles east of Chicago and flying westward at 200 miles per hour. How fast is the distance between the two planes changing when they are equidistant from Chicago?

Another extension of the problem (by analogy to three dimensions) gives rise to the next problem. Three edges x, y and z of a tetrahedron are perpendicular to each other and changing in length. Suppose the initial lengths are $x = 5$ inches, $y = 12$ inches and $z = 17$ inches. If these sides are changing at the rate of one inch per second, $-4/3$ inches per second, and -3 inches per second respectively, what is the rate of change of volume when the three perpendicular faces are all isosceles right triangles?

The foregoing solution was intended to demonstrate, by means of an example, the nature and application of maxims (or heuristics) employed in solving a rate of change problem.

An Example of Mathematical Modeling in a Secondary Classroom

The following is a first example of how one teacher, Paul Foerster, presents applications in a realistic way. Such a process involves models. A model is something that mimics relevant features of a situation that is being studied. For example, a plant collection, a road map, a geological map, a topographical map are all models that mimic different aspects of a portion of the earth's surface. The ultimate test of a model is how well it performs when it is applied to the situation and problem it was constructed to handle. A person on a hiking trip might use each of the above maps for different purposes.

Here we are concerned primarily with mathematical models, that is, models that mimic reality by using the language of mathematics. The following example illustrates important components of the multistage process of mathematical modeling. The emphasis is not on, *given a problem — solve it*, but rather on *given a situation — study it.*

Situation

Teenage drivers (especially boys) are charged considerably more for automobile insurance than adult drivers.

This topic is almost guaranteed to arouse considerable student interest and discussion. In a brainstorming atmosphere many statements (or claims) will be made and numerous questions asked. From this cloudy, fuzzy, ill-formed situation it is possible to generate some fairly precise problems. Examples of such problems might include the following.

1. Are young people really such dangerous drivers?
2. What happens to rates after an accident occurs?
3. Do students who take driver education courses have fewer accidents than other students?
4. Do high school drop-outs have more accidents than high school graduates?
5. How do the insurance rates of parents differ if they include a teenager in their automobile insurance policy?
6. Do middle-aged drivers have fewer accidents than other age groups?
7. How are automobile insurance rates determined?
8. Where would one find some data about age of driver versus number of wrecks?

(Discussion of particular strategies for formulating well-posed problems will be presented in subsequent sections. Also, in the discussion to follow much of the language used follows that used by D. Kerr Jr. and D. Maki in Chapter 1 of *Applications in School Mathematics, 1979 Yearbook*, National Council of Teachers of Mathematics, edited by S. S. Sharon and R. Reys. The purpose in using their terms is to help teachers find an easily obtained reference.)

Devising a well-formed problem from the cloudy situation is perhaps the most significant stage in the modeling process. Kerr and Maki (1979) stress that much modification and simplification go into obtaining a written description of a reasonably precise and succinct problem — which they call a *real model*. For Kerr and Maki, a real model is a problem expressed in real-world terms (hence, the modifier real) but is still a model because not all aspects of the original situation are likely to be incorporated into the problem statement. Let us assume that after many considerations (and trade offs among these considerations) of the situation, the following problem is formulated.

Real Problem or Real Model: "Does the likelihood of having a car wreck depend on the age of the driver and do younger drivers have more accidents?"

This problem does not include the mention of money or amount of insurance rate which was in the original situation — but it clearly relates to that topic. Kerr and Maki state, "After a real model has been formed, the words and concepts of the real model are replaced with mathematical symbols and expressions. The structure that results is a *mathematical model*. The mathematical model deals with mathematical objects (such as sets, numbers, geometric shapes and functions) and with expressions that relate these objects to each other (for example, equations, graphs, transformations and tables)" (p. 1-2).

To mathematize the above problem, students might be asked to sketch a reasonable graph of how the number of wrecks (vertical axis) relates to age of the driver (horizontal axis). Let us assume that after discussion the following graph is presented.

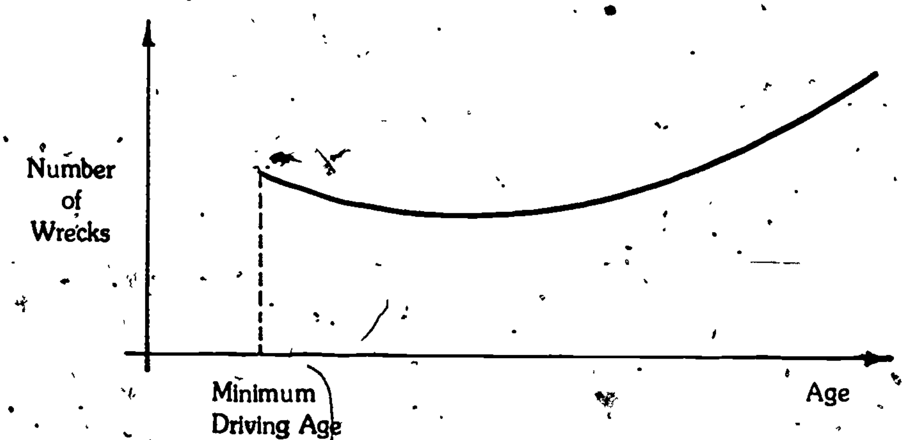


Figure 2.

The graph looks like a *parabola*. Assume, therefore, that a quadratic function will serve as a reasonable mathematical model.

Let t = age of driver in years;

Let W = number of wrecks per 100 million km.

Appropriate units can be determined by seeking out data and noting the units employed in reporting values of W. Thus, the general equation is the following.

$W = at^2 + bt + c.$

To find a, b and c one needs three ordered pairs. Assume that the following data has been obtained.

Data:

t	W
20	440
30	280
40	200

Substitute data into $W = at^2 + bt + c$.

$$400a + 20b + c = 440,$$

$$900a + 30b + c = 280,$$

$$1600a + 40b + c = 200.$$

Solving this system gives

$$a = 0.4,\ b = -36,\ c = 1000.$$

Therefore, the particular equation is

$$W = 0.4t^2 - 36t + 1000.$$

The mathematical model is ready to be used in the next stage of the modeling process.

Conclusions

After a mathematical model is constructed, one applies mathematical tools and techniques to determine *conclusions* based on the model. Next, one tests these conclusions with the real world situation and problem to determine if the model is providing useful information. If the information is not useful, then the stage of the modeling process must be reconsidered. Often one must cycle through the stages many times until an acceptable model (if one exists) is found.

Let us apply the equation

$$W = 0.4t^2 - 36t + 1000$$

to arrive at conclusions for the following three questions.

1. According to the model, who is safer, a 16-year-old driver or a 70-year-old driver?

If $t = 16$, then $W = 526.4$.

If $t = 70$, then $W = 440$.

Conclusion: 70-year-old drivers appear to be safer.

2. What age is safest according to the model?
The vertex of the parabola is at (h,k)

where $h = -\dfrac{b}{2a} = -\dfrac{-36}{0.8} = 45.$

Conclusion: 45 year olds appear to be safest.

3. An insurance agency decided to insure licensed drivers up to the age where the accident rate reaches 830 per million km. What ages would they insure? This problem calls for determining a domain for the quadratic equation

$$0.4t^2 - 36t + 1000 = 830,$$

$$t^2 - 90t + 425 = 0,$$

$$(t - 5)(t - 85) = 0,$$

$$t = 5 \text{ or } t = 85.$$

But this problem specifies licensed drivers. Therefore only one solution of the two positive solutions applies. Conclusion: Domain is $16 \leq t \leq 85$.

At this stage students should discuss how well the conclusions of the mathematical model illuminate the original situation. What predictions can be made based on the model and conclusions? For example, how many wrecks will even the safest drivers be likely to have?

Questions about data should be discussed. Examples: How old? How reliable? In particular, how were values of t and W determined? How do other available values fit with the assumption of a quadratic function? Does the model reflect national trends? Would data on state, county, city or school locale be more important in determining charges for car insurance? What function other than a quadratic might serve as a mathematical model? Should the model be reconstructed based on a different problem which better addresses the original situation? What other related phenomena would likely be modeled using a quadratic equation? For example, how is the average amount paid per accident by insurance companies related to age of driver? Can any decisions or recommendations be forthcoming from this modeling exercise? For example, should students take driver education courses?

A host of other related questions can be raised. A career question might be: What is an actuary? Mathematical questions can be asked about the formula for the x-coordinate of the vertex of $y = ax^2 + bx = c$. For example, what happens when you average the two values of x obtained in applying the quadratic formula? What does the axis of symmetry lie halfway between? What formula yields the y-intercept of the vertex?

Many believe that applied mathematics is the art of model building. Clearly this mode of applying mathematics is quite involved. Further, the traditional vehicle which claims to present applications in mathematics, namely, the word problem of the mathematics textbook provides little exposure to the model building art.

In general, a real application typically begins with a cloudy or ill-formed situation arising from some problem in the real world. Figure 3 presents a model of the multistage modeling process and indicates that the first stage is recognition of some real-world situation.

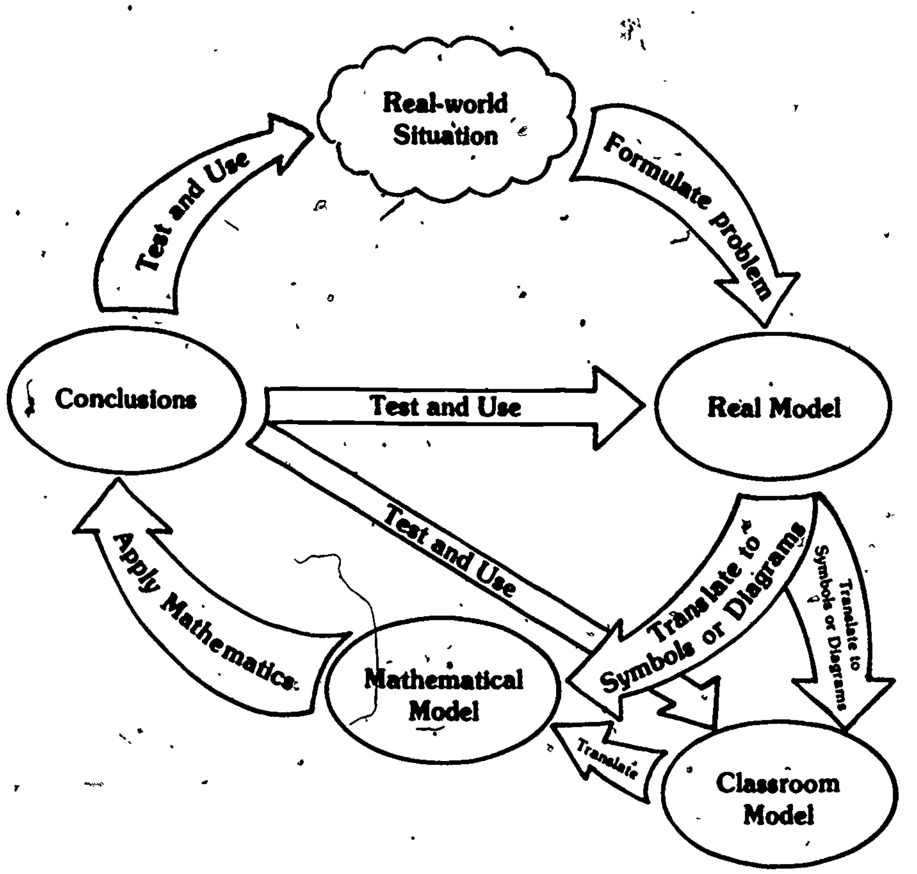


Figure 3. Mathematical model building for the classroom.

After deciding to deal with this cloudy situation, one proceeds to formulate a real-world problem. Such a formulation is usually the crux of the modeling process. A problem which one can reasonably deal with is only obtained after many simplifications and trade offs. Thus, the goal at this stage is to construct a reasonably precise (usually written) problem which incorporates those elements of the original situation which require investigation. As stated earlier the problem constitutes a real model; *real* in that it is usually stated in real-world terms, and *model* in that not all aspects of the original real-world situation are incorporated into the problem.

Replacing the words in the problem or real model with mathematical symbols and expressions leads to the mathematical model. This model often involves equations, inequations, relations, graphs and so forth which interrelate mathematical objects (such as sets, numbers, functions, geometric shapes).

Figure 3 shows that after the mathematical model is constructed, one employs mathematical techniques in order to arrive at conclusions based on the model. Applying mathematics to the model may involve such processes as making deductions, solving equations, estimating parameters, evaluating relationships, analyzing data, simulating processes and making computations.

The conclusions obtained from these mathematical tools and techniques are then tested and compared with the real world to see if the model is providing useful information or insight into the original situation. If it is determined that the model is not yielding useful information then each stage of the modeling process must be reexamined to obtain a viable conclusion. Perhaps the problem needs to be reformulated or a different mathematical model may be constructed. One might obtain a different conclusion if a different approach is used at the solution stage.

In real applications it is often necessary to cycle through the entire process many times before an acceptable model is obtained.

In classroom settings another step is often added to the model-building process. Kerr and Maki state, "In this step the real model is further simplified and put into a setting that will be interesting and comprehensible to students and that will require the mathematics that the teacher wishes to apply. This step results in what we call the *classroom model*" (Kerr and Maki, p. 2). The example presented earlier concerning accidents and age of driver illustrates such a classroom model. The next section presents a collection of classroom models that illustrate the modeling process.

Kirchhoff's Laws

An instance where determining a solution to a system of linear equations leads to solving an important applied problem occurs in the analysis of an electrical network.

Situation

In the construction of electrical circuits it is often necessary to connect several resistances and to analyze the current flow.

Problem

Two laws frequently used in the analysis of electrical circuits are *Kirchhoff's laws*.

1. The algebraic sum of all the currents meeting at a junction is zero. (In other words, all the current flowing into a junction must flow out of it.)

2. The algebraic sum of the R_i (R denotes resistance, i current) around any closed path is equal to the algebraic sum of the voltage in the path.

Consider the circuit in Figure 4 where the batteries

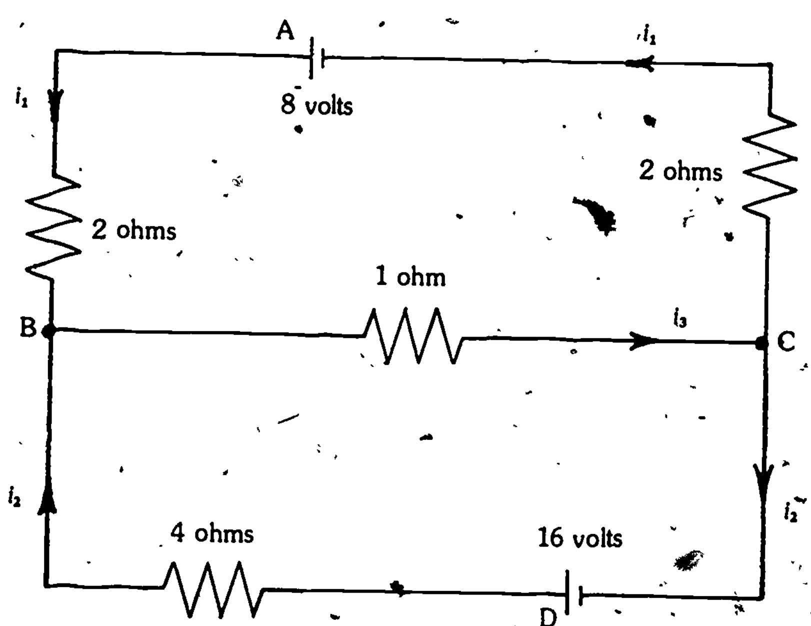


Figure 4.

(denoted —|⊢ are 8 volts and 16 volts; the resistances (denoted ⋀⋀⋀ are one 1 ohm, one 4 ohm and two 2 ohm. How does one determine the currents i_1, i_2 and i_3 which are indicated in the circuit?

Mathematical Model

Assume the current entering each battery will be the same as the current leaving it.

Applying Kirchhoff's first law to each junction,

$$\text{Junction B} \qquad i_1 + i_2 - i_3 = 0,$$

$$\text{Junction C} \qquad i_1 + i_2 - i_3 = 0.$$

Applying Kirchhoff's second law to various paths,

$$\text{Path ABCA} \qquad 2i_1 + 1i_3 + 2i_1 = 8,$$

$$\text{Path DBCD} \qquad 4i_2 + 1i_3 = 16.$$

The problem thus reduces to solving the system of linear equations.

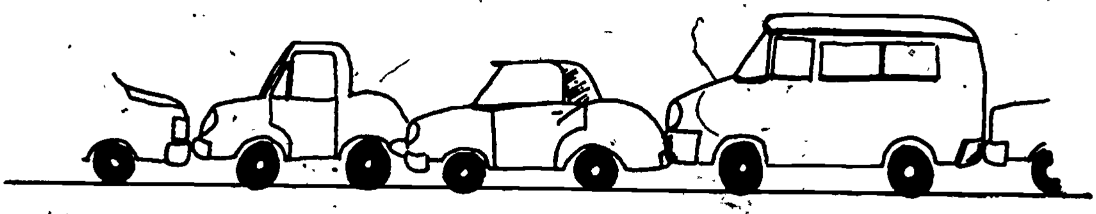
$$\begin{aligned} i_1 + i_2 - i_3 &= 0, \\ 4i_1 + i_3 &= 8, \\ 4i_2 + i_3 &= 16. \end{aligned}$$

Conclusion

The solution of the above system is $i_1 = 1$, $i_2 = 3$ and $i_3 = 4$. The units here are amperes. The solution is unique, as is to be expected in the analysis of such a situation. A negative current would imply that the flow was actually in the direction opposite to the one initially assumed. Of course, in practice, electrical networks can involve many resistances and the problem involves solving a large system of linear equations.

Was it noticed that the applying Kirchhoff's laws lead to having four equations in three unknowns? However, the system was such that a unique solution was obtained. Is this an unusual or usual happening when applying Kirchhoff's laws? Who was G. R. Kirchhoff? (He published his laws in 1847 while at the University of Königsberg and invented the concept of *tree* which is so important in network analysis.)

Traffic Flow During Peak Periods



The concepts and tools of network analysis have been found to be very useful in such areas as information theory, operations research, management science and computer science. The following situation deals with an application of network ideas in the study of transportation systems. The conclusion shows how a system of linear equations with many solutions often arises in real-world problems.

Situation

How can one carry out an analysis of traffic flow through a road network during peak periods?

Problem

To simplify the situation assume that all the streets are one-way, and the streets form a grid (or rectangular) pattern. To be definite, consider the city road network in Figure 5. The arrows indicate the direction of traffic flow. The figure also shows the traffic flow in cars per hour for the peak evening period. For example,

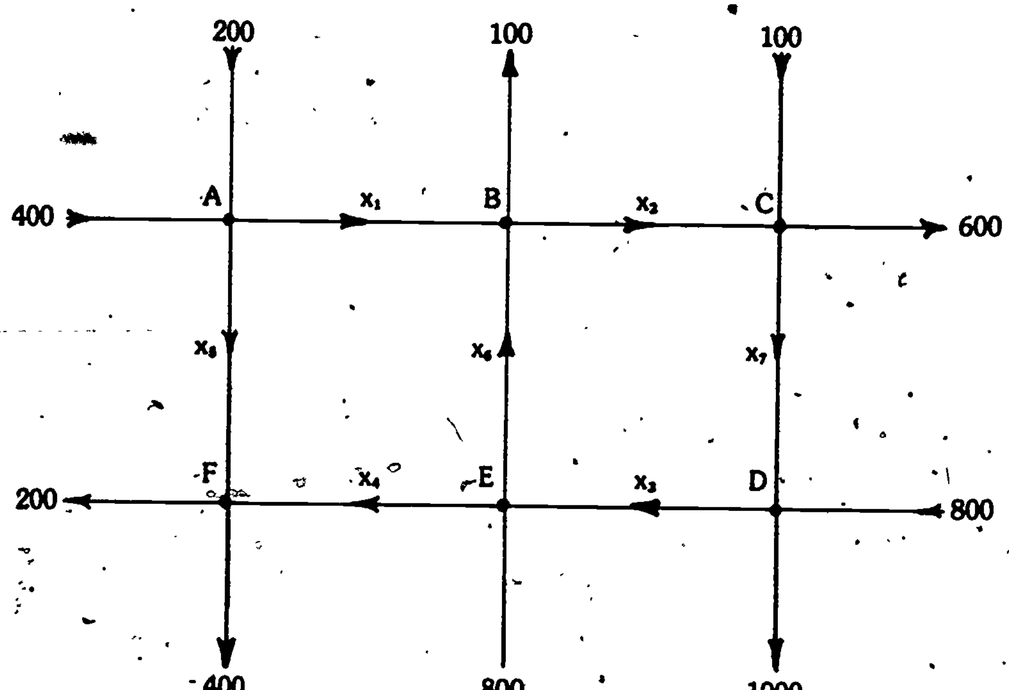


Figure 5.

400 plus 200 cars per hour enter junction A. Our task is to construct a mathematical model that describes the traffic flow in all branches. It is reasonable to assume that all traffic entering a junction must leave that junction if there is no congestion.

Mathematical Model

Let $x_1, x_2, \ldots, x_7$ denote the traffic flow along the various branches. Thus x_1 is number of cars per hour that travel from junction A to junction B during the peak period. (See Figure 5.) The assumption that all traffic entering a junction must leave that junction is called the *conservation flow constraint*. (Compare it with the first of Kirchhoff's networks.) This constraint leads to the following system of linear equations.

$$
\begin{array}{lcccccccl}
\text{Junction A} & x_1 & & & & +x_5 & & & = 600 \\
\text{Junction B} & x_1 & -x_2 & & & & +x_6 & & = 100 \\
\text{Junction C} & & x_2 & & & & & -x_7 & = 500 \\
\text{Junction D} & & & -x_3 & & & & +x_7 & = 200 \\
\text{Junction E} & & & x_3 & -x_4 & & -x_6 & & = -800 \\
\text{Junction F} & & & & x_4 & +-x_5 & & & = 600
\end{array}
$$

The first equation above is easily obtained after students note that x_1 and x_5 flow away from junction A while $400 + 200 = 600$ cars flow into junction A. Therefore, by the conservation flow constraint $x_1 + x_5 = 600$. The equation for junction E derives from $800 + x_3 = x_4 + x_6$.

Conclusions

Even if students do not know a method for dealing with large numbers of equations (such as Gaussian elimination), it is quite easy — using substitution and addition/subtraction techniques — to express, say, each of x_1, x_2, x_3, x_4 and x_5 in terms of x_6 and x_7. (Again, a main point of this problem is to indicate that solutions to systems of equations which model the real world are not always unique.)

By adding the second equation (junction B) to the third equation (junction C) we have

$$x_1 = 600 - x_6 + x_7.$$

From the third equation we see

$$x_2 = 500 + x_7.$$

From the fourth equation we obtain

$$x_3 = -200 + x_7.$$

From the fifth equation we have

$$\begin{aligned} x_4 &= 800 - x_6 + x_3 \\ &= 800 - x_6 + (-200 + x_7) \\ &= 600 - x_6 + x_7. \end{aligned}$$

From the sixth equation we obtain

$$\begin{aligned} x_5 &= 600 - x_4 \\ &= 600 - (600 - x_6 + x_7) \\ &= x_6 - x_7. \end{aligned}$$

In summary, we have the following.

$$\begin{aligned} x_1 &= -x_6 + x_7 + 600 \\ x_2 &= x_7 + 500 \\ x_3 &= x_7 - 200 \\ x_4 &= -x_6 + x_7 + 600 \\ x_5 &= x_6 - x_7 \end{aligned}$$

It is interesting to note that $x_1 = x_4$. Of course that could have been obtained by merely subtracting the last equation from the first in the original equations.

A typical use of the solution set could be as follows. Suppose road CD was scheduled for repair or widening or some other work. Then it would be desirable to minimize the flow x_7 of traffic along CD. Assume flows along other branches can be controlled or latered in some way to handle the effect of less traffic on CD, e.g., through traffic light patterns at other junctions or through use of fewer available lanes on CD.

In order to consider how to minimize the flow x_7 examine $x_3 = x_7 - 200$. Since all flows must be greater than or equal to zero the equation implies that the minimum flow for x_7 is 200, for otherwise x_3 could become negative if $x_7 < 200$. (A negative flow can be interpreted as traffic moving in the direction opposite to the one allowed on the one-way street.) Thus road work must permit a flow of at least 200 cars per hour on the branch DC in the peak period.

What is the effect on the other branches if $x_7 = 200$?

Then
$$\begin{aligned} x_1 &= -x_6 + 800, \\ x_2 &= 700, \\ x_3 &= 0, \\ x_4 &= -x_6 + 800 \end{aligned}$$
and
$$x_5 = x_6 - 200.$$

Thus to obtain a minimum flow $x_7 = 200$ one must set $x_3 = 0$, i.e., close ED and $x_2 = 700$. The remaining flows are not uniquely determined. If we set $x_6 = 300$, say, then a feasible set of flows would be $x_1 = 500$, $x_4 = 500$ and $x_5 = 100$. Make a sketch of the road network and determine if these values of $x_1, x_2, \ldots, x_7$ satisfy the conservation flow constraint. Note that, in this case, $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 2300$ cars. Is this the maximal number of cars that can be handled during the peak period? As a project, a student (or the teacher) could give a report on the famous Ford and Fulkerson *Max Flow-Min Cut Theorem* which treats maximal flows through networks — using many of the concepts presented above.

A Simplified Classroom Model of the Peak Period Traffic Flow Situation

Situation

This situation is identical to the previous one only that it is simpler because it deals with fewer junctions, branches, flows and equations. It would be used either in the classroom or as a homework problem.

Problem

Using the ideas presented in the previous section, *Traffic Flows During Peak Periods*, construct a mathematical model that describes the traffic network in Figure 6.

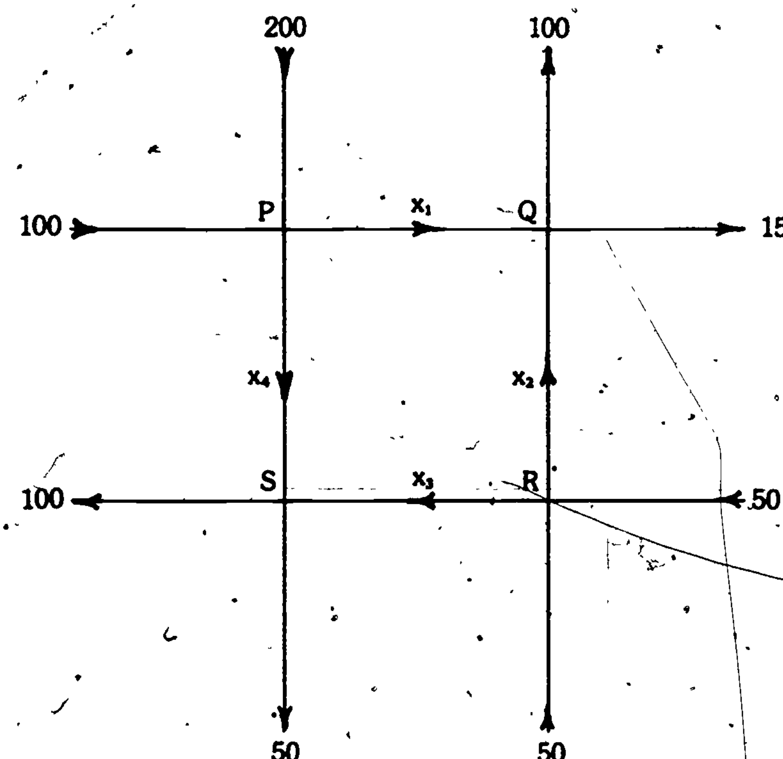


Figure 6.

Mathematical Model

Let x_1, x_2, x_3 and x_4 denote flows as shown in Figure 6. Applying the conservation of flow constraint we have the following system of equations.

Junction P	x_1	+					x_4	= 300	(1)
Junction Q	x_1	+	x_2					= 250	(2)
Junction R			x_2	+	x_3			= 100	(3)
Junction S					x_3	+	x_4	= 150	(4)

Conclusions

A solution in terms of, say, x_1 is as follows:

$$x_2 = x_1 + 250 \qquad (5)$$

$$x_3 = x_1 - 150 \qquad (6)$$

$$x_4 = x_1 + 300 \qquad (7)$$

Equation (5) comes from equation (2) above, equation (7) from equation (1). To obtain (6) we merely note from (3) and (5) that

$$x_3 = 100 - x_2 = 100 - (-x_1 + 250)$$
$$= x_1 - 150.$$

What if road work was required for PQ with flow x_1. That is, if we wanted to minimize x_1, equation (6) shows that the smallest possible value for x_1 is 150. If $x_1 = 150$, then $x_2 = 100$, $x_3 = 0$ (i.e., close RS), and $x_4 = 150$.

If we wanted to maximize x_1, then $x_1 = 250$ since junction Q can only absorb that number of cars in this model. To obtain this value for x_1 we would have to close QR so that, from (5), $x_2 = 0$. Also if $x_1 = 250$, then from (6) and (7), respectively, we have $x_3 = 100$ and $x_4 = 50$.

Thus if we assign some value to x_1 the other flows are determined. For example, letting $x_1 = 200$, yields $x_2 = 50$, $x_3 = 50$ and $x_4 = 100$. Note that in each of the above three instances (i.e., when $x_1 = 150$, $x_1 = 250$, $x_1 = 200$), that the total volume of traffic was 400. Examination of equations (1) and (3) shows that $x_1 + x_2 + x_3 + x_4 = 400$. Can we conclude that the maximal flow is in fact 400?

Numerous projects can be developed around the idea of traffic flow. What are some *traffic* problems within a school, e.g., hallway, cafeteria. Is there a street intersection which, in your opinion, should have a traffic light? This last question is a golden opportunity for students to gather data. (A stopwatch would be helpful.) Traffic engineers can be consulted. Many factors such as safety, costs, convenience and legal constraints are soon encountered. Of course, air traffic continues to be a challenging situation involving such concerns as air-traffic control, airport location and air safety.

An Engineering Application

The following problem illustrates how a problem in engineering reduces to determining the solution to a system of linear equations.

Situation

In designing a structure such as a bridge, one must know the forces that will come into play when the structure is subjected to various loads. Often the structure is composed of bars connected together to form a rigid framework. The bars are usually pin-pointed at their ends.

Problem

Assume that since weights of the bars are usually insignificant when compared to those of the forces they carry that they will be neglected in analyzing the distribution of forces over the bars.

Assume, also, that each bar is in equilibrium under the forces at its two ends. Hence, these two factors acting along the bar are equal and opposite. The bar itself exerts equal and opposite forces at its end joints. They may both be outward or both inward. The force in the bar is called a *stress*.

Consider the framework in Figure 7, which carries a load of, say, 20 tons at its upper joint, denoted by A. How can one determine the stresses in the bars?

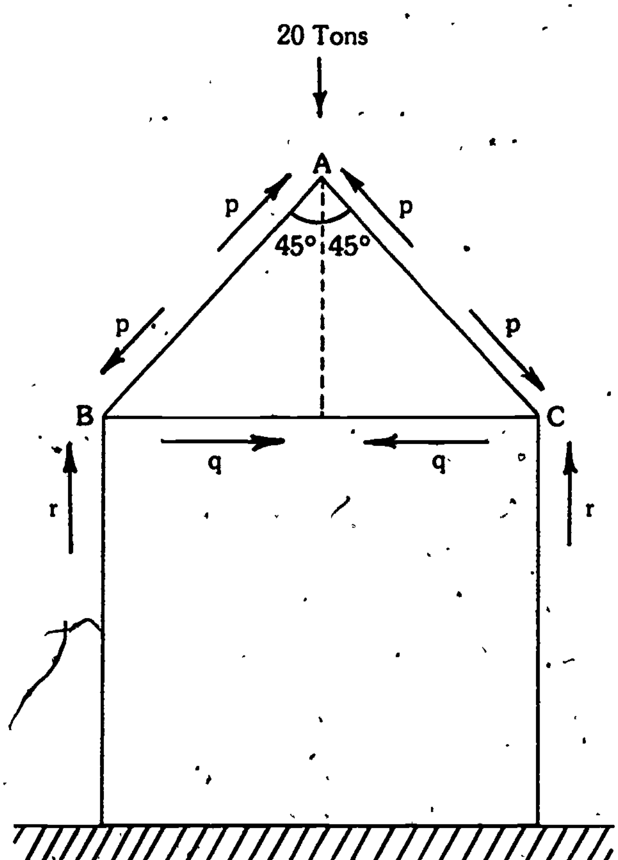


Figure 7.

Mathematical Model

Consider the joint A. Intuitively, one can see that the forces exerted by the two bars at this joint must be upward in order to balance the 20 tons if the framework is going to remain in equilibrium. By the symmetry of the situation, the forces exerted by the bars are equal. Label these forces p. The forces at B and C are labelled p, q and r as shown in the figure. The student should be able to see that these bars either push or pull at the joints. At joint A, since its in equilibrium, the 20 tons must be equal to the sum of the components of the ps in the vertical direction. The component of p in the vertical direction is

$$p \cos 45°, \text{ or } \frac{p}{\sqrt{2}}.$$

The component of a force in a certain direction is that force multiplied by the cosine of the relevant angle. (If students have not studied vectors, this problem or some similar problem, provides an opportunity to discuss analysis of forces via simple examples.) Hence, for joint A,

$$\frac{p}{\sqrt{2}} + \frac{p}{\sqrt{2}} = 20.$$

Now examine joint B. Forces in the vertical direction give $p \cos 45° = r$, or

$$\frac{p}{\sqrt{2}} - r = 0.$$

Forces in the horizontal direction give $p \cos 45° = q$, or

$$\frac{p}{\sqrt{2}} - q = 0.$$

Hence the problem reduces to solving the system of linear equations.

$$\begin{aligned} p &= 10\sqrt{2} \\ \frac{p}{\sqrt{2}} - r &= 0 \\ \frac{p}{\sqrt{2}} - q &= 0 \end{aligned}$$

Conclusions

The system has the solution $p = 10\sqrt{2}$, $r = 10$, $q = 10$.

If a negative number for any stress had been obtained that would imply that the stress is actually in the direction opposite to the one assumed. Thus the correct initial choice of directions is not crucial to the final analysis.

The actual stresses in the bars themselves are necessarily equal in magnitude and in a direction opposite to the forces exerted by the bars on the joints. The stresses in the bars are represented in Figure 8.

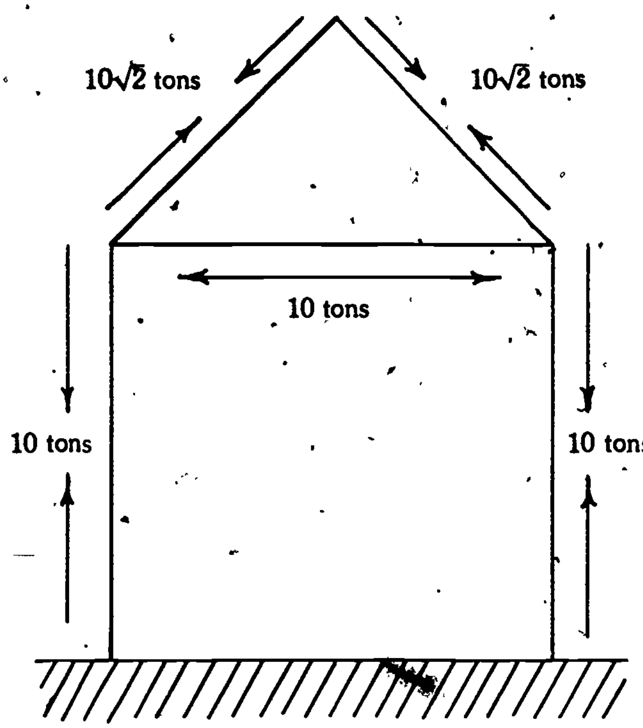


Figure 8.

Football Pools and Dumb Gamblers

Situation

Football pools are one of the most popular forms of gambling. One type of pool is the following. A list of football games is printed on a ticket. For each game the team judged to be weakest is given sufficient points to make that game a toss-up. For example, if Team A is a 15-point favorite over Team B, then Team A must win by over 15 points to be judged a winner in the pool.

Problem

A player is trying to decide between the following two options.

Option 1. For \$1, a player selects four teams he expects will win from among the ten games printed on the ticket. Thus six of the games are ignored. If the player's four selections turn out to be winners, \$10 is returned to the player resulting in a \$9 profit; otherwise, the dollar bet is lost.

Option 2. For \$1, the player tries to pick all ten winners. In this case, \$150 is returned if the player succeeds in correctly choosing all winners; \$20 is returned if nine of the ten winners are selected. In all other cases the dollar bet is lost!

Which of the two options, the four game or the ten game, is better for the player?

Mathematical Model

Assume that for each game selected the player has a $\frac{1}{2}$ chance of being a winner. Also assume that each game played is independent of the other games. For simplicity assume that ties are not allowed.

Option A is considered better than option B if the average profit or *expectation* which a player can expect is greater for option A than option B. Let E(4 game) denote this expected average profit for the 4-game option. Then, it is easy to show that $E(4\ game) = 9(1/16) + (-1)(15/16) = -6/16 = -0.375$.

Thus the player loses an average of 37.5 cents for every dollar bet.

For the second option one notes that the probability of picking ten winners is $(\frac{1}{2})^{10}$ or $\frac{1}{2^{10}}$, and the probability of picking nine winners is $10/2^{10}$.

(*Remark*. The probability of selecting one winner is $\frac{1}{2}$. Because it is assumed that games are independent then the probability of picking 10 winners is $(\frac{1}{2})^{10}$. To select exactly 9 winners out of 10 games is the same as the number of ways one can select exactly one loser, which is the combination of 10 games taken 1 at a time or 10. Thus the probability of selecting exactly 9 winners is $10/2^{10}$. The probability of selecting exactly 8 winners is $\binom{10}{2}/2^{10} = 45/2^{10}$.)

Thus, the player's expectation E(10 game), i.e., the second option, is

$$E = 149(1/2^{10}) + 19(10/2^{10}) + (-1)(1 - 1/2^{10}) = -\frac{337}{512} = -0.658.$$

Thus, employing the second option, the player loses an average of 65.8 cents per play.

Conclusions

Between options 1 and 2, it turns out that option 1 is much better. Of course in both cases — as with all gambling situations — the expectation is negative. (The house always has the edge!)

Comparison of football pools with other forms of gambling, e.g., craps or roulette, shows that, overall, pools represent a very poor form of gambling. (Note: We ignored ties. In practice, if there is a tie, the player loses. So the situation actually is worse than indicated above.)

Another popular form of gambling is the Numbers Racket where one attempts to select a 3-digit number. If $1 is bet, then the payoff, say $700, leads to the following.

$$E(\text{select 3 digit number}) = 699\left(\frac{1}{1000}\right) + (-1)\left(\frac{999}{1000}\right) = -0.30$$

This is a very insidious situation which often preys on low income earners.

Good resources for classroom discussions of gambling are the vice squad of a police department and other enforcement agencies.

The next collections of problems are included because they deal with interesting situations and they are fun to do.

Problems for Students

The Lion and the Unicorn Problem

In chapter three of Lewis Carroll's *Through the Looking-Glass* one finds Alice wandering around the Forest of Forgetfullness, where she is unable to remember the days of the week. Assume she meets a Lion, who lies on Monday, Tuesday and Wednesday, and a Unicorn, who lies on Thursday, Friday and Saturday. All other times both animals tell the truth. The Lion states, "Yesterday was one of my lying days." The Unicorn says, "Yesterday was one of my lying days too." How can you help Alice deduce the day of the week?

Hint: Consider what days each animal could make their statements if they were telling the truth or if they were lying.

Solution

The Lion can say, "I lied yesterday" only on two days: Monday (if he lied) and Thursday (if he told the truth). Similarly, the Unicorn can make the same statement only on Thursday and Sunday. Thus Thursday is the only day on which both the Lion and the Unicorn can make the statement.

The "Shoot the Gap" Problem

The following incident occurred in North Carolina. An automobile was traveling at a high rate of speed on a level highway when it encountered a 25-foot wide washout. The driver tried to "shoot the gap" but missed by one foot. (See Figure 9.) A law enforcement agency examined this accident. One of their questions was, should the driver get a speeding ticket?

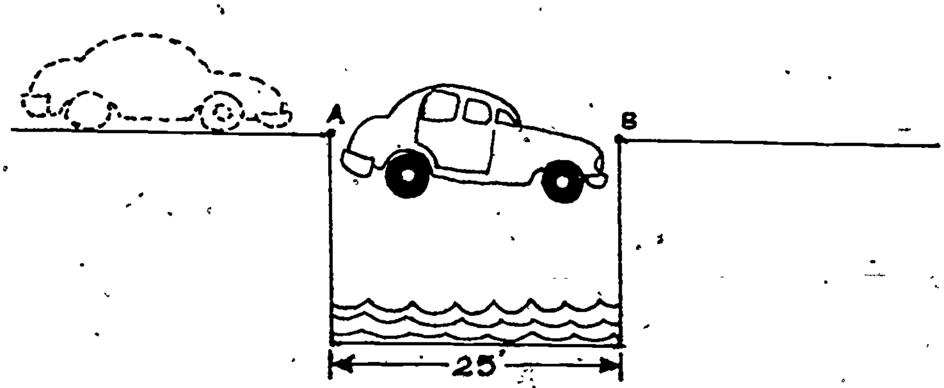


Figure 9.

They approached a professor who made the following analysis: During the period when the car travelled from one edge (point A) of the washout to the other bank (point B), it fell one foot. Because of this, the professor assumed the car can be treated as a freely falling body in order to determine the time required to drop this distance. The formula for the distance(s) a freely falling body drops in time (t) is $s = \frac{1}{2}gt^2$. Substituting 1 foot for s and 32 feet per second per second for g yields

$$s = \frac{1}{2} gt^2, \qquad \text{Thus} \quad t^2 = \frac{1}{16} \text{ and}$$

$$1 = \left(\frac{1}{2}\right)(32)t^2; \qquad t = \frac{1}{4}.$$

Hence, the car was freely falling for ¼ seconds. But since the car travelled a distance of 25 feet we have

$$d = rt$$

$$25 = r\left(\frac{1}{4}\right) \text{ or}$$

$$r = 100 \text{ feet per second.}$$

To find r in terms of miles per hour the professor did the following.

$$100 \frac{\text{ft.}}{\text{sec.}} \times 60 \frac{\text{sec.}}{\text{min.}} \times 60 \frac{\text{min.}}{\text{hour}} \times 1 \frac{\text{mile}}{5280 \text{ ft.}} = 68.\overline{18} \text{ miles per hour}$$

To check his work, he recalled that 60 mph is equivalent to 88 fps, and then solved the following proportion.

$$\frac{60}{80} = \frac{x}{100}$$

$$\text{or} \quad x = 68.\overline{18} \text{ miles per hour}$$

Since the speed of the auto exceeded the federal speed limit of 55 mph, the professor advised the law enforcement agency to issue a speeding ticket to the driver of the car.

Questions

1. What simplifying assumptions were made in the analysis of this problem?

2. If the washout had not occurred on a level highway but through a hill, what minimum speed would guarantee a successful "jump" (a la Evel Knievel) if the angle θ (see Figure 10 below) were known?

3. Conversely, if the jump was successful at a given speed, what is the value of θ?

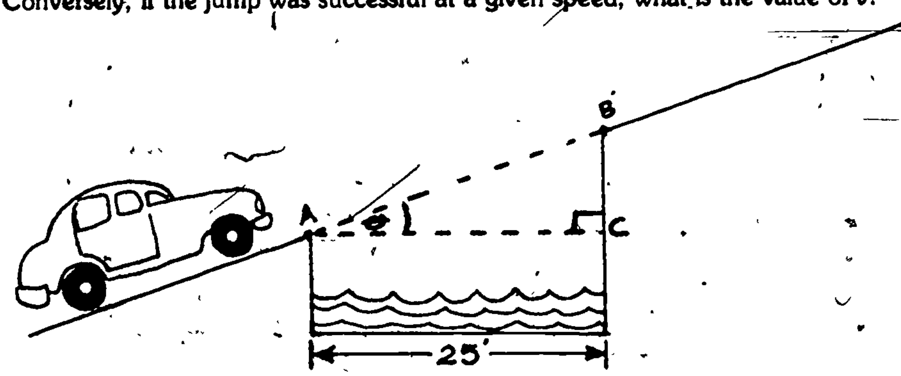


Figure 10.

The Impossible Problem

Can one solve the impossible problem? Two numbers (not necessarily different) are chosen from the range of positive integers greater than one and not greater than 20. Only the sum of the two numbers is given to mathematician S. Only the product of the two is given to mathematician P.

On the telephone S says to P, "I see no way you can determine my sum."
An hour later P calls back to say, "I know your sum."
Later S calls P again to report, "Now I know your product."
What are the two numbers?

(See *Scientific American*, December, 1979 for a discussion of this amazing problem.)

Knight and Knave Problem

A logician comes upon two men on an island inhabited only by *knights*, who always tell the truth, and *knaves* who always lie. The logician asks one of them, "Are either of you knights?" When the man — call him X — responds, the logician instantly knows the answer to her question.

a. Is X a knight or a knave?
b. What is the other man?

Hint: The key to solving the above problem lies in the fact that X's answer enabled the logician to find the solution.

Solution

If X's response had been yes, the logician would not have learned anything. (If X is a knight, one or both of the men could be knights, and if X is a knave, both of them could be knaves.) Thus X must have said no.

Now if X were a knight he would have had to say yes, but because he said no he must be a knave. Further, since he is a knave his no must be false, thus at least one of the two men must be a knight. Therefore, the other man is a knight.

An Insightful Solution

Situation

Perhaps one of the most gratifying experiences for a teacher occurs when a student devises a truly insightful approach to a problem which yields an elegant solution. An example of such insightful thinking was employed with the following:

Problem

The usual tic-tac-toe game is played on a two-dimensional grid configuration. This game is won by placing either three **X**s or three **O**s in a row. There are eight three-in-a-row lines on the configuration which may yield a win, i.e., three horizontal lines, three vertical lines and two diagonal lines. Additionally, there is a commercially available three-dimensional tic-tac-toe game which is played on a $4 \times 4 \times 4$ cube. How many four-in-a-row lines are there through the cube by which this game may be won?

Mathematical Solution

There are a variety of ways to attack this counting problem — most of which are quite complicated and involve many cases. A brilliant approach employed by young Leo Moses was essentially the following: Consider a $6 \times 6 \times 6$ cube which encases the given $4 \times 4 \times 4$ cube with a shell of unit thickness. If one extends in two directions a winning line in the inner $4 \times 4 \times 4$ cube to the outer shell this line pierces two of the unit cubes in the shell. Also each unit cube in the shell is pierced by only one winning line. Thus each winning line corresponds to a unique pair of unit cubes in the outer shell, and the number of winning lines is simply one-half the number of unit cubes in the shell, namely

$$\frac{6^3 - 4^3}{2} = \frac{216 - 64}{2} = 76.$$

This approach can be generalized to show the number of winning lines for a cube of edge k in n-dimensional space is

$$\frac{(k + 2)^n - k^n}{2}$$

Conclusions

A teaching strategy which may be used to aid students in *discovering* the above approach is to have them place a border of unit squares around a 3×3 grid corresponding to the usual tic-tac-toe game. If a student determines that each 3×3 winning line, when extended in two directions on the 5×5 grid, corresponds to a unique pair of unit squares in the outer border, then it follows that the number of winning lines is one-half the number of unit squares in the border, namely

$$\frac{5^2 - 3^2}{2} = \frac{25 - 9}{2} = 8.$$

Such work with two-dimensional cases has led some students to examining what occurs when one encases a three-dimensional cube with a shell of unit thickness.

The Ferry Boats Problem

Two ferry boats travel back and forth across a river at constant speeds. Without any loss of time, they turn at the river banks. When leaving opposite sides of the river at the same time, they meet for the first time 700 feet from one side of the river. They continue on their way to the banks, return and meet a second time 400 feet from the opposite shore. Determine the width of the river.

Solution

When the boats first meet, the total distance both boats have traveled is just the width of the river. By the time they meet again, the total distance the boats have traveled is three times the width of the river. Since the boats are traveling at the same speed, their second meeting occurs after a total time that is three times as long as the times for their first meeting. The first boat, ferry A, traveled 700 feet to get to the first meeting. Ferry A would go 2100 feet or three times as long to get to the second meeting. When making the second meeting, A goes all the way across the river and back 400 feet. The river must be $2100 - 400 = 1700$ feet wide.

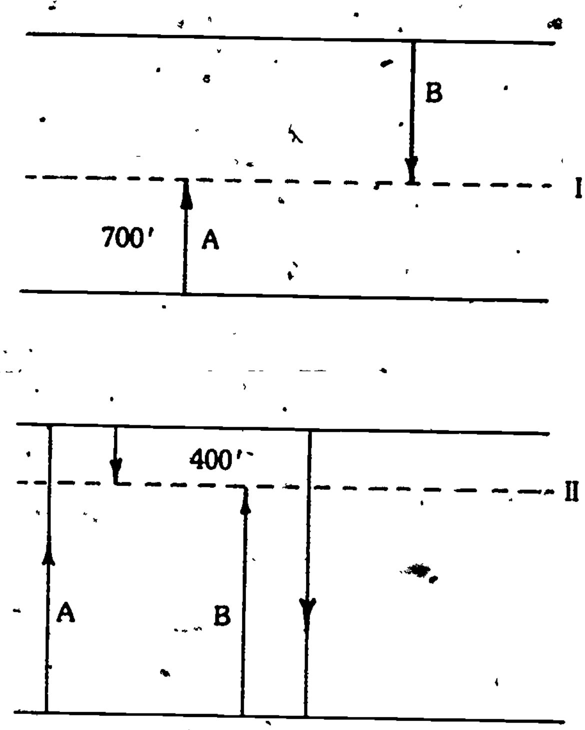
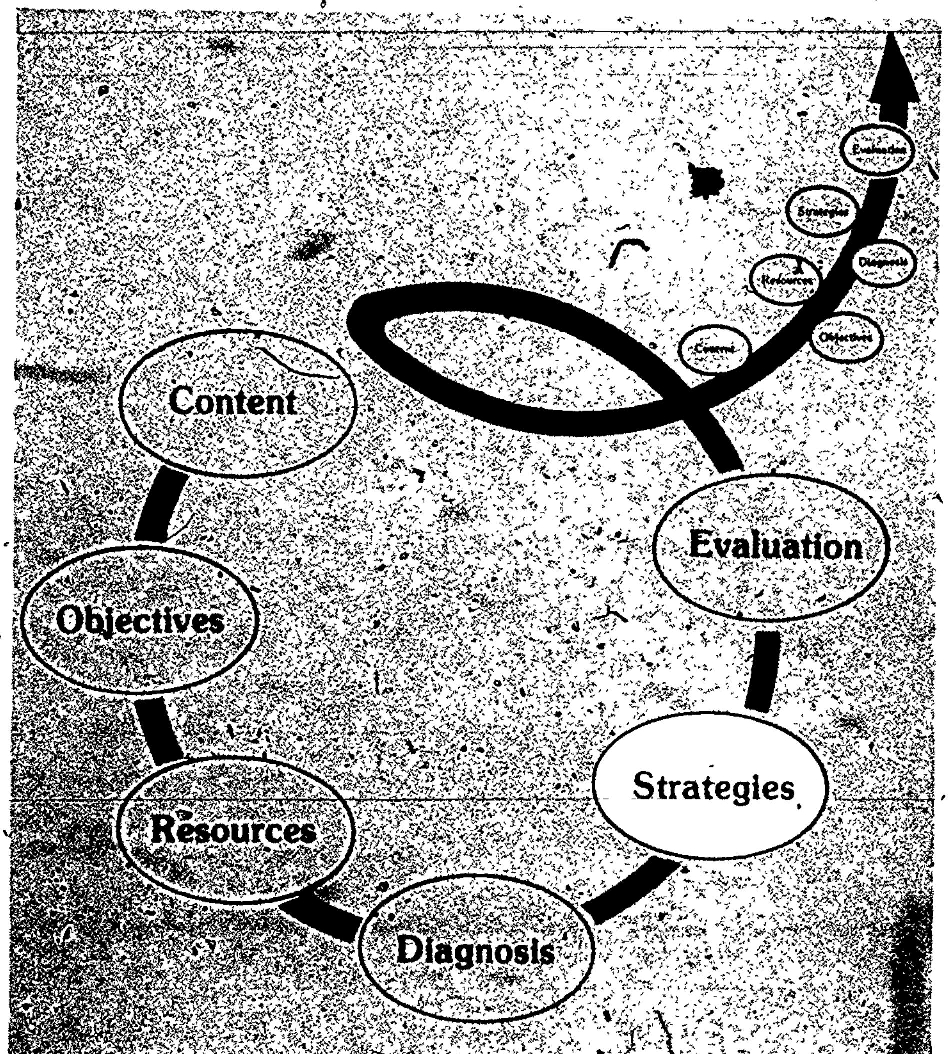


Figure 11.


Evaluation
Strategies
Diagnosis
Resources
Objectives
Content
Content
Objectives
Resources
Diagnosis
Strategies
Evaluation

STRATEGIES

Strategies for Helping Students Learn Mathematics

One of the fundamental goals of all mathematics teachers should be to teach students how to learn mathematics effectively and efficiently. This is a difficult task since mathematics is unique in several respects and requires approaches different from other subjects. The statement, "No one can teach a student mathematics, but the student may learn mathematics," is a meaningful insight to keep in mind. Even though students hear, read or see the words or symbols of a lesson these may remain mere words or symbols unless the students are able to internalize them. Students need to develop skills that will allow them to actively participate in the learning process. The discussion which follows provides practical suggestions for helping students learn how to listen, read, study and communicate mathematics.

Learning by Listening

Since classroom instruction involves a great deal of verbal interaction, it is important for students to develop good communication skills. Active participation is the key element. This requires students to listen to discussions, ask questions, try to discover relationships, anticipate the next step, take notes (i.e., identify important ideas and examples) and reflect on the topic being discussed. To help students improve their listening skills, teachers must seek to involve students and encourage their participation. It is for this reason that verbal interaction and questioning are often considered the heart of teaching. Suggestions are provided in the following sections that should prove helpful.

Providing Motivation

One key ingredient to active involvement is getting students interested in the topic. Some teachers fall into a habit of starting a discussion in the same way over and over. "Yesterday we studied . . ." "Today we are going to study . . ." "Open your text to page . . ." "Notice that today's lesson involves . . ." Teachers should use a variety of techniques to begin the discussion such as games, problem situations, puzzles, applications, discovery situations (How could you find how many without counting?) or a challenge (Can you find a technique that always works?). A card file or notebook of ideas for introducing lessons on various topics may be helpful. Sources for these ideas may be other textbooks, resource books or professional journals such as *The Mathematics Teacher.*

Provoking Student Interaction

Establishing an accepting climate will encourage student questions. Opportunities should be provided for students to react to each other's responses. (For instance, "John, do you think Mary's approach will always work?") It is very important for teachers to exhibit that they listen and value student responses.

Developing Good Questioning Techniques

Good teachers ask good questions and provide opportunities for their students to ponder and respond to the questions. However, many teachers fall into the habit of asking questions that require merely recall or recognition and even tend to answer, themselves, many of the questions posed. It is important to concentrate on asking higher level questions that require students to interpret the information and explain, analyze and apply or search for solutions in both structured and unstructured situations. Learning to be patient is equally important. Patience prevents one from answering the question immediately or interrupting or interpreting a student's response before any other student has an opportunity to comment. It is important that teachers foster an atmosphere that encourages student interaction and questions. Unfortunately, the emphasis in schools on extrinsic rewards such as grades or praise often detract student curiosity and interaction. Good questioning techniques can help develop intrinsic motivation.

Providing Opportunities for Summary and Reflection

It is important for teachers to provide time for students to discuss ideas they have studied. This should aid the students in integrating the previously learned ideas into the topics currently being studied. Summarizing and reviewing helps students put in perspective the content they are studying. It helps students grasp why they are learning certain content, why it is important and where it will be used.

Learning Through Reading

Since words and symbols are the essential elements of the language of mathematics, students must learn to read and use this language. If students cannot read mathematics with understanding, they will be severely handicapped in any further study of mathematics and in their daily lives. Simple life skills involve reading and interpreting charts and graphs in materials such as newspapers, travel schedules (bus, train and plane), recipes in cookbooks, training manuals and many others.

Every mathematics teacher has the responsibility for teaching students to read mathematics. Research indicates that student achievement is improved if the reading skills instruction is integrated with the mathematics instruction. The unique aspects of mathematics present several significant problems that require special attention. Mathematics text materials tend to be written precisely and in a compressed and concise style. The vocabulary is highly specialized and a large number of symbols, abbreviations and notations are used. The ideas related to the study of mathematics are sometimes abstract. There is often a sequence of steps involved in a computational procedure or problem-solving process that requires the student to deviate from the typical left-to-right convention used in other types of reading. This involves reading (decoding) right-to-left (as in some computations) and vertically (as in graphs and charts). All of these factors emphasize the importance of helping students learn to read mathematics.

Helping Students Read Mathematics

The way in which material is read depends on the purpose for reading. At times students need to read material rapidly (or skim) to obtain an overview. Skimming allows the reader to recognize the important ideas that are being presented and to understand the organization of these ideas. It is important for students to note the new terms, phrases and concepts that are being presented. Skimming also helps students recognize knowledges or skills that may need to be reviewed. For example, a student skimming a chapter in an algebra textbook that introduces algebraic expressions and equations might make the following list of topics, terms and questions:

1. What are algebraic expressions?
   a. Numbers, variables and signs of operations
   b. What is a variable?
2. What are the steps in evaluating an algebraic expression?
   a. Order of operations
   b. Set of replacement values
   c. Powers of a variable
   d. Exponential notation
3. How can true, false and open equations be recognized?
   a. Equations
   b. Open equations
   c. Solution of an equation
   d. Solve an equation
   e. Equivalent equations

After the student has skimmed the material and recognizes the organizational pattern and new terms to be studied, then the student must *reread* the beginning section much slower. It should be emphasized that speed is not the major concern when reading mathematics. It is important for the student to read with comprehension. This type of reading requires concentration and attention to detail. Overlooking a portion of a symbolic expression or word may result in misunderstandings.

Thus, it is important to incorporate techniques for helping students with their reading problems. Many of the common techniques used by reading teachers are quite helpful. These techniques include the following.

1. Have students read the selection silently and prepare to discuss the questions.
2. Provide written questions that help students focus on the important ideas being presented.
3. Help students recognize the new vocabulary and symbols that they will encounter in their reading.
4. Provide explanations of how the new material is related to previously learned material.
5. Help students with troublesome situations such as graphs, tables, figures or notations.

Developing Specific Skills

The reading of mathematics requires several specific skills that are somewhat different from those taught in typical reading or language arts programs. For example, the reading of charts, diagrams, tables, graphs and measuring instruments require special instruction. The important point is to devote explicit attention to teaching reading in the mathematics classroom. Many of the techniques that prove helpful are extensions of the techniques used by reading teachers.

- Developing vocabulary

It is important to make usre that words and symbols convey meaning. Most textbooks are organized to provide help with the specialized vocabulary of mathematics. This information is usually found in the table of contents, the glossary and the explanatory materials. Most texts also use special techniques such as italics, underlining, additional color or other highlights to emphasize important ideas or new terms. Teachers should help students learn to use these features effectively. However, this alone will not solve the problem because many common words take on different meanings when used in mathematical situations. The list below (taken from McKillip, Cooney, Davis and Wilson, 1978, p. 190) gives examples of common words that have different mathematical meanings.

acute	clock	factor	negative
add	closed	foot	odd
alternate	column	greater	opposite
altitude	common	intercept	origin
angle(s)	commute	interior	perfect
array	compass	intersect	place
associate	complement	intersection	plane
axes	concave	invert	plot
balance	convex	irrational	point
bar	correspond	lateral	power
base	count	law	prime
between	cross	leg	product
borrowing	curve	less	property
boundary	degree	like	radical
braces	~~distance~~	lowest	rational
cancel	~~distribute~~	major	ray
cardinal	divide	map	real
carrying	element	mean	right
casting	even	minor	root
check	exterior	mixed	round
chord	face	natural	row
ruler	similar	space	union
scale	simple closed	square	unit
second	simple form	term	volume
set	solution	twin primes	yard
sign			

There is some evidence that poor readers tend to give only the common meaning to words even when used in a mathematical situation. Often the meaning of a word varies according to the context in which it is used.

The movie will appear during *prime* time.
Find the *prime* factors.
Equivalent fractions; *equivalent* expressions.
Equivalent sets; logically *equivalent*.

It is important to emphasize the different interpretations of words when they are used in different contexts. Even though the words may have similar meanings in a general context, their meanings may be quite different in specific contexts. This can be confusing and students need help in determining these similarities and differences.

Other helpful techniques for developing vocabulary include using root words and prefixes and noting words or phrases that have similar meanings.

Use root words to analyze meaning.

multi-	*multi*plication, *multi*plier, *multi*ple,
divi-	*divi*sion, *divi*dend, *divi*sor,
equi-	*equi*distant, *equi*angular, *equi*valent.

Use prefixes to analyze meaning.

un-	*un*equal, *un*known, *un*defined,
in-	*in*equality, *in*finite, *in*center,
poly-	*poly*gon, *poly*hedron, *poly*gonal,
non-	*non*intersecting, *non*negative, *non*linear.

Recognize different forms of the word.
factor, factorization, factoring
bisect, bisector, bisecting

- Developing skill with symbols and abbreviations

Mathematicians have developed a universal language of symbols. This language of symbols is often interspersed with words and abbreviations. Students need help in learning to understand and use this shorthand. Exercises that involve matching symbols or abbreviations to words or phrases may be a useful technique to use in teaching. See the following for example.

Match the word with the symbol.

Multiply	=
Centimeter	%
Percent	$\perp$
Perpendicular	hr.
Hour	+
Add	×
Equals	cm
Millimeter	mm

It should also be emphasized that symbols can be used in a variety of ways. For example, the numerals 2 and 3 can have very different meanings depending on what other symbols or organizational patterns are used — for instance, 23, 32, 2^3, 3^2, $\sqrt[3]{2}$, 2/3, 2×3, $3 + 2$, 3.2, $2x^3$ . . . Students need help in recognizing and interpreting the specialized use of symbols in mathematics.

The discussion above suggests that mathematics teachers need to devote greater attention to the problem of helping students learn to read mathematics. Many mathematics teachers have had limited training in this area. However, there are reading specialists and language arts teachers in every school system who are valuable resources.

Learning by Studying

Two of the major goals for all mathematics classes should be to help students (a) learn the mathematics content thoroughly and (b) develop the study skills that will allow them to learn mathematics independently. Learning is a lifelong process and the skills developed during the formal schooling process should provide the tools necessary for students to add to their knowledge and understanding of mathematics after they leave the classroom.

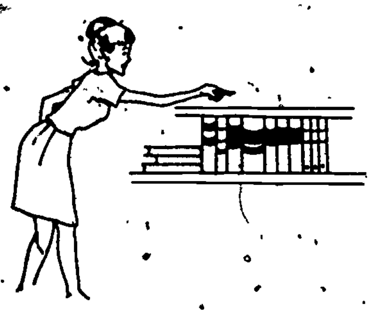
The study of mathematics involves the development of accurate applications of skills and manipulations in certain formal processes and an understanding of the meaning of these manipulations as well as the underlying concepts so that they can be applied to problem-solving situations.

In the study of mathematics most students tend to rely too much on memory. Competence in mathematics is not a matter of just remembering facts, formulas and methods of manipulations. Algebra students may memorize the quadratic formula and be able to use it to find the solutions to quadratic equations during a course. However, several years later most students will have forgotten the formula even though they may remember the general appearance of the formula such as, ". . . it involves $-b \pm \sqrt{\ }$ and something about 2a." Could the students find the solutions to $2x^2 + 9x - 5 = 21$? Obviously they could find the formula in a textbook. Thus, memorizing the formula was a convenience in the algebra class. But students could have completed the square or solved the equation by factoring if they had an understanding of the process. Most students were exposed to the derivation of the quadratic formula by completing the square. But understanding the process and reasons for completing the square are much more important ideas.

In helping students learn how to study mathematics, it is important to keep in mind just what it is that students should learn and what will be useful and can be transferred. Obviously, there are certain facts, skills and processes that students should remember. Otherwise, learning is inefficient. One task of the teacher is to help students acquire information effectively and efficiently so that the higher goals of understanding and problem solving can be accomplished. The suggestions provided below should help students develop good study skills.

Motivating Interest

Psychologists agree that to learn something thoroughly, students must have an interest in the material being studied. Students do not remember material in which they are not interested or do not see as important. One of the primary roles of a teacher is that of motivator. Mathematics teachers need to be enthusiastic about the study of mathematics and they need to convey the beauty and application of their subject.



Helping Students Select What They Study

There is a great deal of detail provided in textbooks and students are not always sure what they should learn. Students need help in selecting what they study. Outlines should be provided that contain an emphasis of the major ideas, facts or processes that students should learn. Teachers need to review these major ideas frequently.

Establishing an Intention to Remember

Students need to realize that they will be required to remember certain aspects of the material. This focuses their attention on the task and increases the probability that they will strive for understanding or learn the ideas as they are presented.

Providing Review of Prerequisites

The understanding of new material is based on the foundation of accumulated knowledges and experiences. Teachers must be sensitive to the prerequisite skills and concepts invovled in the new material. Otherwise, the new material may not be meaningful.

Practicing Meaningful Organization

Students need to be able to cluster facts and ideas and to put them together in some meaningful fashion if they are to learn ideas efficiently. Teachers must carefully plan lessons and assignments with some organizational scheme in mind.

Providing Opportunities for Recitation

Recitation helps students transfer material from short-term memory to long-term memory. Recitation involves saying (orally or mentally) the ideas to be remembered. For example, students can cover words or phrases and say the definition or explain the process to themselves. Some teachers help students with this process by having them make notebooks or study sheets. An example is shown below.

Pythagorean Theorem	relation for right triangles $a^2 + b^2 = c^2$
Rhombus	quadrilateral (equilateral) all properties of parallelogram diagonals are perpendicular diagonals bisect angles

Providing Opportunities for Consolidation

Helping students consolidate the information and ideas that are presented is a valuable learning activity. Consolidation can be stimulated by summary or review sessions or summary study guides. It may, however, involve the process of recitation or writing the ideas or facts read or studied. As students recite or write they are holding each idea in their minds for four or five seconds which helps transfer the information into more permanent storage or memory.

Maintaining and Distributing Practice

Research studies have indicated the importance of distributed practice. Distributed practice or maintenance of previously covered material normally involves short periods of class or individual study time usually not greater than 10 minutes. Students should recognize that it is important to study material in short review sessions after it is well organized rather than try to use longer periods of mass practice. For example, some students depend too heavily on studying the night before a major exam rather than using distributed study periods to help them learn the material more thoroughly.

In general, students who are willing to exert a sincere effort can learn to study and use mathematics effectively. The student who develops good study habits and a sincere interest in mathematics will continue to learn and enjoy mathematics. Students who develop poor study habits and never really understand mathematics or its uses will probably not retain much of what they have been taught. Thus, the success of mathematics teachers must be judged by the degree to which they become less important to their students' learning of mathematics.

Strategies for Teaching Mathematics

The teaching of mathematics is a challenging and exciting endeavor filled with delights, discouragements, pleasures and difficulties. The real satisfaction of teaching mathematics comes from teaching in a creative way a fascinating subject to individuals who are important.

Most teachers of mathematics would include in their definitions of the teaching process the act of transmitting knowledge, thinking skills, attitudes, understandings and problem-solving skills. Explaining exactly how this transfer occurs is a difficult task.

This section presents a modified flow chart of the common elements of the teaching process. The complexity of the teaching and learning processes and the variation among teachers and students imply that teaching and learning are highly individualized and personalized endeavors. Thus, the improvement of teacher effectiveness is a responsibility that must be assumed by the teacher. The willingness to try different teaching strategies and to critically analyze the effectiveness of these strategies should be an integral part of the professional development of every mathematics teacher.

The Teaching Process

For the purpose of this discussion, teaching strategies will be broadly interpreted to mean approaches, methods or activities that might be employed in the teaching of mathematics. The flow chart below provides an overview of the teaching process.

Mathematics Content

- Identify content using sources such as local and state curriculum guides, textbooks and professional references.
- Select mathematics content. This is an important step and should not be treated lightly. Notice the differences among the objectives in collections C, P and E of this guide.

Learning Objectives

- Have in mind the specific behaviors that students are to demonstrate or exhibit.
- Formulate both cognitive and affective objectives.

Cognitive	**Affective**
Knowledge (facts and skills)	Beliefs
Understanding (concepts and generalizations)	Attitudes
Problem solving (application and proof)	Appreciations
	Values

(Continued Next Page)

Learning Resources

- Make use of available resources such as other textbooks, supplemental commercial materials or models, media (such as films, overhead projector, tape recorder), calculators, computer, library materials, newspaper, catalogs, professional journals, other students and community resources personnel.
- Learning resources in the mathematics classroom have too long been limited to chalk, chalkboard and textbook.

Diagnosis and Preassessment

- Determine the prerequisite skills and concepts for the new topic.
- Assess students' readiness to learn the new topic through previous tests, quizzes, homework or a short written or oral quiz. Assessment may also be done through teacher observation, games or a more formal approach such as pretest or diagnostic tests which may be available with instructional materials.
- Provide review for areas of weaknesses where needed.

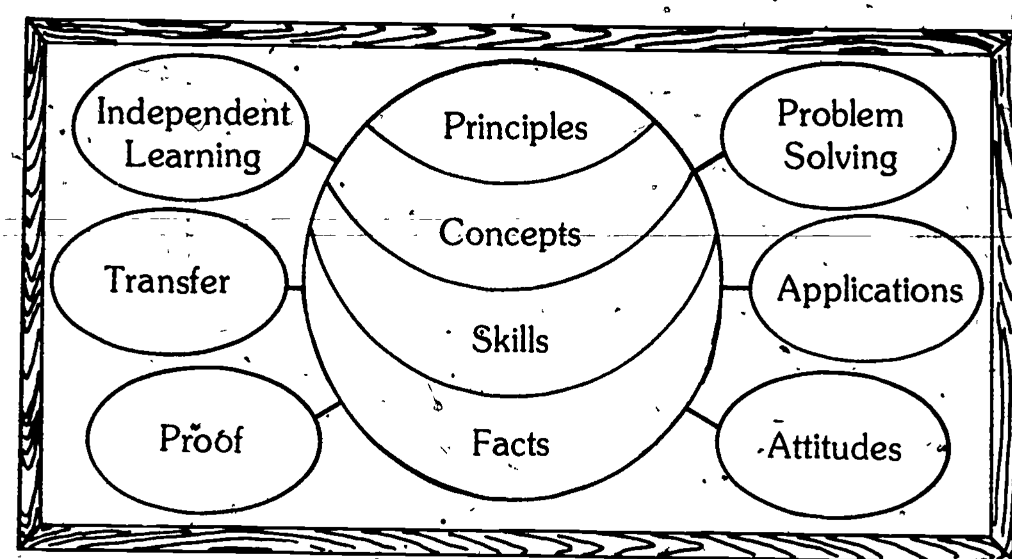
Teaching Strategies

- Select teaching strategies that reflect a variety of variables such as the nature of the mathematical ideas, expected levels of performance and the developmental levels of the students in the class.
- Basic elements of mathematics such as facts, skills, concepts and problem-solving procedures are learned in different ways; thus, the teaching strategies for each of these elements should accommodate these differences.

Evaluation

- Assess how well each student has learned the new material.
- Determine which objectives have been adequately mastered and which objectives need additional reinforcement.
- Evaluate the effectiveness of the teaching strategies and materials used in the lesson.
- Plan for specific periods of review of important aspects such as skills, facts and concepts.

The diagram below illustrates some of the common elements of mathematics instruction for which teachers must find strategies.



Identification of Strategies

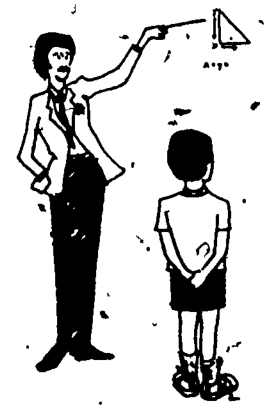
If a group of educators was asked to make a list of commonly used strategies for teaching mathematics, the list would probably include the following. Lecture, discussion, inductive, deductive, synthetic, analytic, informal, discovery, explanatory heuristic and guided discovery. The basic components of these strategies are considered below under the two broad headings, heuristic or discovery and expository.

Heuristic or Discovery

Discovery, heuristic or inquiry strategies are among the oldest known teaching strategies and date back to the time of Socrates. Even though discovery or guided discovery approaches have been much acclaimed, there is no generally agreed-upon description of this method. The guided discovery approach requires that students investigate some topic and formulate an hypothesis, a conjecture or a procedure, test the reasonableness of their work or formulate a conclusion. Inquiry is guided by teaching through questions and formulation of a model of the situation.

The most common objective of a guided discovery lesson or series of lessons is to direct students to discover a concept, generalization, algorithm or process. However, some educators and psychologists suggest that the process involved in discovery leads to a generalized way of thinking or attacking problems.

A guided discovery lesson typically begins with a class discussion of particular questions or a problem situation. The teacher directs the discussion by asking questions and directing student questions to other students. This process can be very exciting and thought-provoking; however, it can also be time-consuming. This is one of the major criticisms of discovery approaches. Obviously, not every topic lends itself to a guided discovery approach. However, this strategy provides an excellent opportunity for eliciting participation, increasing motivation and promoting learning.



Expository

The expository strategy is the most commonly used method for teaching mathematics. The expository strategy involves explanation, interpretation and demonstration. This approach often includes short lectures or explanations, demonstration of skills or procedures, teacher-student discussion (questions), individual or small group work on problems or examples and follow-up in the form of homework. The following are hints concerning the expository strategy.

- Discuss the objectives of the lesson with the students.
- Identify and discuss prerequisites.
- Know the content of the lesson well. If the primary focus is on presenting the mathematical ideas, it is difficult to be attentive to student reactions.
- Be a good observer and ask timely questions. A teacher should not ask rhetorical questions such as, "Do you understand?"
- Develop a sensitive and accepting attitude that encourages questions and participation.
- Anticipate problem areas and be sensitive to mistakes or misunderstandings so that they can be used constructively.
- Be sure to use a variety of examples, illustrations and nonexamples.
- Provide opportunities for practice and application.
- Evaluate the learning.

Variations of the expository strategy described above often involve greater student participation. These approaches are typically more analytic and involve more learning resources. A modified expository strategy might involve more demonstrations and discussion. For example, in a geometry lesson on constructions, the teacher may depend more on demonstration, experimentation and discussion.

Variations of the expository strategy can also involve the use of materials such as demonstrative models, manipulatives, calculators and computers. This type of variation has often been referred to as a laboratory approach.

Even though the expository strategy has received criticism in recent years due to its teacher-dominated, group-oriented posture, it still remains one of the most useful and practical strategies available. However, it takes a master teacher to make full use of its potential.

Special Considerations for Teaching for Concept Development

Concepts are abstract ideas which are used to classify events and objects. They are understandings of what something is or is not. Some concepts are associated with names or labels. The teaching of concepts requires that students understand the meaning of these terms or labels.

Concepts are constructed from experiences and thoughts and are formed over time, even over years. For example, the concepts of function or equivalence class take years for many students to understand.

Teaching for concept development requires careful, thoughtful presentation of the material. It is not always obvious that students have grasped the essential aspects of a new concept. Generally, teachers expect students to choose and recognize examples of the concept or exhibit examples of the concept. Notions about concepts need to be communicated. Hence, students must be able to interpret and use both the word and symbol for the concepts.

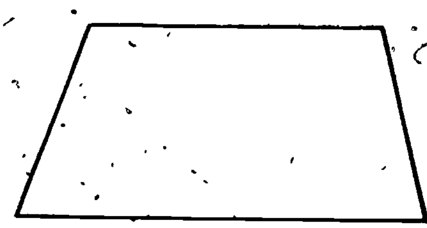
Why not just have students learn to recite the definition? General definitions can help with concept formation; however, a rote recitation of a definition does not assure the presence of the essential aspects of the idea. Also, there is a danger in over-relying on verbalizations. For example, students may recite the definition of a function or a trapezoid and yet have only a vague notion of the concept.

It is essential that teachers provide a variety of opportunities for students to give or choose examples of the concepts. This provides feedback about an individual student's understanding of the concept.

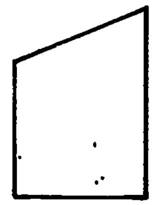
The Role of Examples and Nonexamples

Although it is not clear how many examples or the variety of examples are required to teach a new concept, it is safe to assume that concepts involving greater levels of abstraction will require more examples. For instance, the concept of a regular polygon would require more illustration than the concept of a triangle.

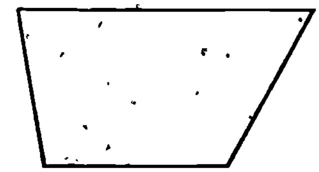
Students often focus on irrelevant properties in examples. For instance, a student being introduced to the concept of a trapezoid may observe that the trapezoid in the example below has four sides, top and bottom sides parallel; no congruent sides, no right angles; and the base is the largest side.



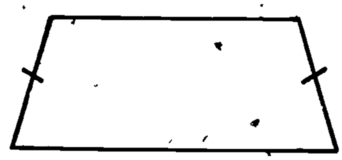
Note the irrelevant properties below.



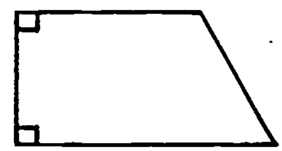
Top and bottom not necessarily parallel



Bottom not the longest side



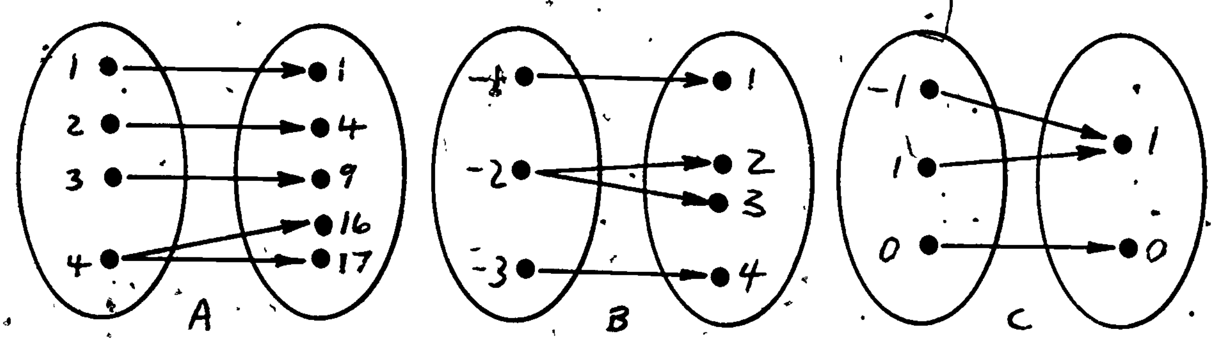
Two congruent sides



Two right angles

Using nonexamples helps students to focus on concept-relevant properties by noting their absences. Nonexamples should be intorduced after students have experienced some examples. Choose nonexamples which closely resemble the examples presented.

Which of the following is a function from the first set to the second set?

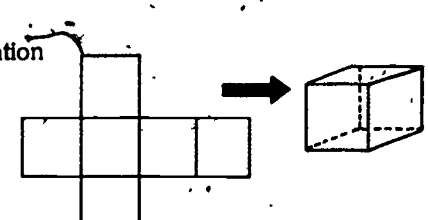


The Role of a Variety of Representations

Concepts can be represented in concrete, pictorial or visual (semiabstract), and symbol or verbal (abstract) modes. It is very important to begin with concrete presentations. Where possible, actual physical objects or representations should be used. *Concrete* seems to be a relative term and can vary depending upon the student's background. Visual representations are a very powerful tool and often are not exploited to their maximum.

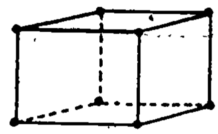
Concept: Polyhedra — Cube

Concrete representation



cardboard model folded to make cube

Visual representation



Abstract representation

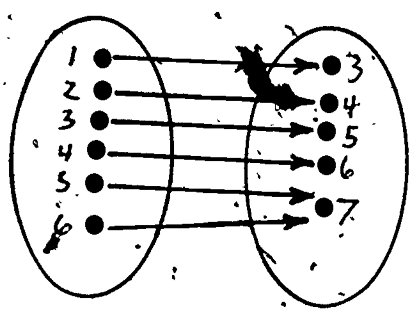
Cube ABCDEFGH

Beyond the common categories of representational modes, it is important to realize that many concepts can be represented in a variety of ways. Experiencing and visualizing concepts in a variety of settings increases the likelihood that students will develop a firm grasp of the essential aspects of the concepts.

Consider the following representations for the same function from set x to set y (domain contains specified x values only).

A: $\{(1,3),(2,4),(3,5),(4,6),(5,7),(6,7)\}$ ORDERED PAIRS

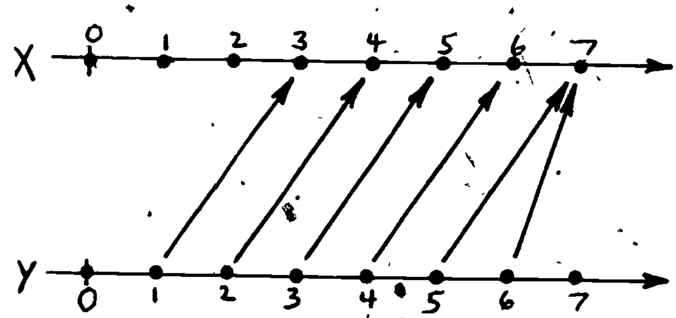
B.



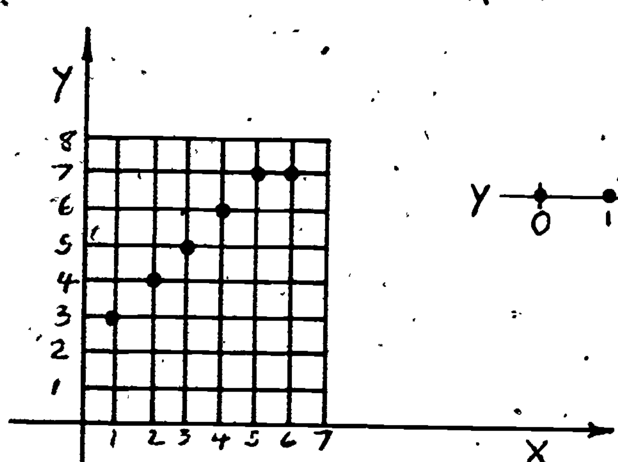
MAPPING DIAGRAM
SET TO SET

C. $y = \begin{cases} x+2 & \text{if } x = 1,2,3,4,5 \\ 7 & \text{if } x = 6 \end{cases}$

FORMULA



MAPPING DIAGRAM
LINE TO LINE



GRAPH

Which representation is most obviously a function?

Explain why you made that particular choice.

In which representation are the domain and range most obvious?

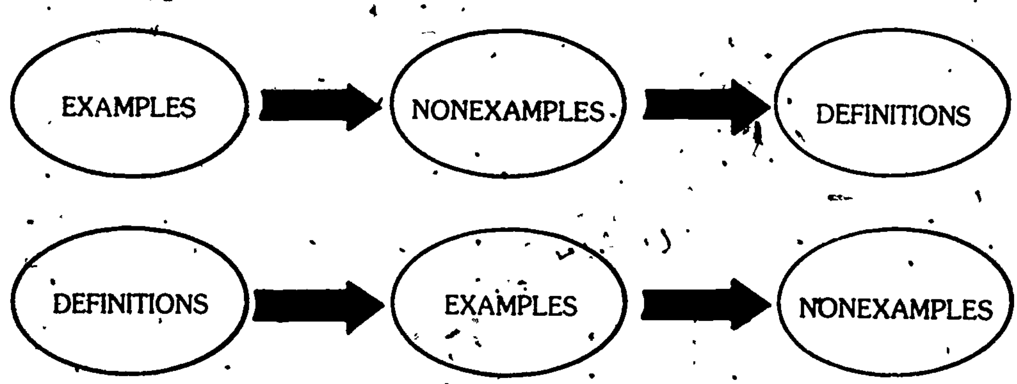
Find the: Domain
Range
Image of 3
The element which has 6 as its image

The Role of Definitions

Particular care needs to be given to the role of definitions in concept development. Often emphasis is placed too early on formal definitions. Many students get "lost" with words and symbols. The meaning that students actually attach to formal definitions is not what most teachers assume. Teachers need to ensure that students understand the words and discuss their meanings. Studying regular polygons several weeks ago does not guarantee that students will remember the concept. Teachers often fall into the trap of asking the class, "How many of you remember what a regular polygon is?" and a few hands or nods are sufficient to continue the lesson. It would be a relatively simple task to ask the students to draw three regular polygons on their papers. This type of activity provides the feedback that teachers need to assess the students' levels of understanding and takes only a few minutes of class time. While walking around the room, the teacher can quickly assess the general level of understanding and comment on any misconceptions that are observed.

The teacher must make every effort to discuss all parts of a definition and to clarify any confusing aspects. For example, the definition may include a troublesome phrase such as "at least two parallel sides." It is often this attention to details, anticipating and clarifying, that becomes the crucial factor. It is helpful to present both examples and nonexamples of definitions as they are discussed.

In recent years, research efforts have been devoted to identifying several types of *moves* that teachers use in teaching concepts (Henderson, 1970; Cooney, Davis, & Henderson, 1975). To date, the research evidence does not suggest a *best* way; however, it is important to experiment with various moves that prove to be successful in your classroom.



Teaching concepts is one of the most difficult and important aspects of mathematics teaching. Particular care should be given to how concepts are presented to students.

Special Considerations for Teaching Facts and Skills

Facts and skills require that students recognize or recall specific statements, conventions or procedures and require automatic responses. Imitation followed by repetition or practice is the key to acquiring a skill. However, these conditions do not ensure that the learning is efficient. This section presents suggestions for teaching skills more effectively.

Developing Meaning Before Practice

Many secondary students are somewhat vague about the meaning of certain numbers and symbols such as 2/3, 0.333 . . ., $\sqrt{3}$, $2x^3$, $|x|$ with which they come in contact because these ideas are remote from their everyday experiences. It is difficult for students to develop skills with operations or procedures involving numbers or symbols that they only vaguely understand. Research studies indicate that students learn skills better by spending less time on practice and more time on developmental activities [Ship and Deer (1960), Zahn (1966), Shuster and Tigge (1965)]. These developmental activities should stress meaning and understanding through work with concrete or visual representations, laboratory activities, teacher demonstration or group discussion. In general, material that has meaning is learned faster and retained longer than material that is learned by rote. It is helpful for students to realize the importance of learning the skill or process. Teachers need to provide meaningful examples of where the skill, fact or process will be used.

Planning Maintenance and Review

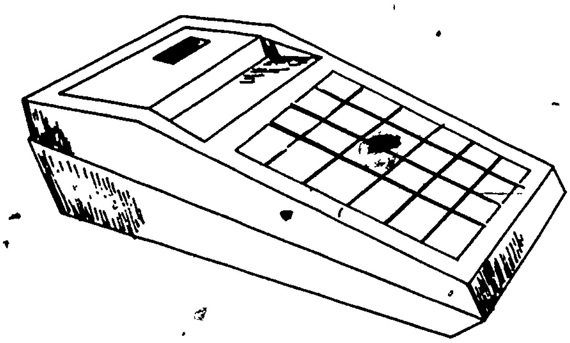
The students should become involved as soon as the algorithm or skill has been demonstrated. The students may vocalize or demonstrate the steps involved. Every student should have the opportunity to try a problem. The answers should be provided for the first few problems so that immediate feedback will be available. If the teacher walks around the class and looks for students who need help, errors may be avoided and frustration reduced.

Positive student behaviors should be reinforced and students should be helped to recognize mistakes. This will help students assume the responsibility of evaluating their own work. Few students make *careless mistakes*. Most student errors are the result of an imbedded misconception and require in-depth analysis on the part of the teacher in order to correct.

After these initial lessons, plans for distributed periods of practice are necessary. The forms of the practice should be varied, such as short worksheets, games, overhead projector skills and tape recorded drills.

Further review and maintenance should be provided as new topics are introduced. Many teachers use review problems as warm-up activities while they are performing routine tasks such as taking the roll or handing out papers. Other teachers find flash cards or overhead drills involving mental arithmetic or paper and pencil to work effectively.

Strategies for Using the Hand-held Calculator in Mathematics Instruction



These suggestions reflect use of the calculator as a tool for learning mathematics. Although techniques for using the calculator must be taught, these techniques are not ends in themselves, but rather a means to an end. The objective is for the learner to comprehend mathematics sufficiently to become an effective and efficient user of mathematics.

In mathematics the hand-held calculator performs the role of efficient calculating and — since it is used by the student rather than the instructor — is a learning tool. In this role as a learning tool the results of its calculations are not the ends for which it is used.

The calculator is an instructional aid, and — as with any other aid — the student must be familiar with its capabilities. Prior to formal use of the calculator, students need to investigate their calculators. Examples of questions that might be considered are the following.

What is the logic system of the calculator?

How many digits are capable in the display?

What are the basic functions on the calculator?

Is there addition of and multiplication by a constant capability?

How does the % key work in finding the percent of a number and in calculating a discount of sales tax?

What are the "clear" and "clear entry" capabilities?

There are two major strategies for use of the calculator in the mathematics classroom. In one, the student uses the calculator as an instructor. This is most useful with those students for whom the goal is still the mastery of paper and pencil calculation, and for those students in algebra who are having difficulty with algebraic principles and procedures. In the other strategy, the student plays the role of instructor to the calculator, making decisions and telling the calculator what to do. It is in this respect that the calculator excels above all other instruments and demonstrates its power and value in mathematics learning. In the next two sections explanations and examples are given of these two strategies.

There are two major strategies for use of the calculator in the mathematics classroom. In one, the student uses the calculator as an instructor. This is most useful with those students for whom the goal is still the mastery of paper and pencil calculation, and for those students in algebra who are having difficulty with algebraic principles and procedures. In the other strategy, the student plays the role of instructor to the calculator, making decisions and telling the calculator what to do.

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The Calculator as Instructor

Of course the hand calculators will not explain to students the long division process or any other calculation procedure. With the exception of some specialty models, the calculator will not even generate practice problems for the student to do. But the calculator can be used effectively for verification. It is an inexpensive and ever-handy monitor of numerical calculation, one for which the student will not have to wait because it is busy helping another.

Mastery of Paper and Pencil Calculation

The calculator will provide immediate response to students who have had instruction in the operations and who may attempt to answer questions such as the following.

1. Was 2.973 correctly subtracted from 19.84?
2. Where does the decimal point go when multiplying 1.973 by 0.24?
3. Has 0.164 been correctly converted to rational (a/b) form? (The calculator will quickly convert the answer back to decimal form).
4. What is 29% of $42.00?
5. Is $8 - (-3) = 11$, or -5, or maybe 5?
6. Is $-\frac{1}{2} = \frac{1}{2}$ or $-\frac{1}{2}$?

Might students merely use calculators to find the solutions to the problems to begin with and make no effort to solve the problems by paper and pencil methods? Of course they might. But if they do, then mastery of paper and pencil calculation is not *the* goal, and the students are not likely to master those paper and pencil calculations even without the calculator. With the calculator at least they produce the correct answers and perhaps they will receive some insight into the paper and pencil procedures they have been shown for so many years.

Acquisition of Elementary Number and Operation Facts

For high school students who do not have elementary number and operation facts the calculator will produce those facts for self-paced review without the need of books or tables or note sheets. Some examples of such operation facts are

1. the sums of all pairs of digits
2. the multiplication table
3. the difference of all pairs of digits
4. the division table
5. the squares of the integers from 1 through 25
6. the powers of 2 up through the 10th
7. the powers of 3 up through the 6th
8. the cubes of the integers from 1 through 10

With the calculator students will be ready to explore many mathematical ideas and make discoveries of patterns and relationships for which paper and pencil calculation is too tedious and time consuming. The calculator can be used to explore number patterns, to discover relationships, to practice mental estimation, to reinforce the inverse relationships of operations and to aid in problem solving.

Examples for Exploring Number Patterns

1. Use your calculator to find the decimal names for the fractions listed.

$\frac{1}{9}$ $\frac{2}{9}$ $\frac{3}{9}$

$\frac{1}{11}$ $\frac{2}{11}$ $\frac{3}{11}$

$\frac{1}{7}$ $\frac{2}{7}$ $\frac{3}{7}$

Use what you found to predict these.

$\frac{5}{9}$ $\frac{7}{9}$ $\frac{8}{9}$

$\frac{6}{11}$ $\frac{9}{11}$ $\frac{10}{11}$

$\frac{4}{7}$ $\frac{5}{7}$ $\frac{6}{7}$

Describe what pattern you found for 9ths, 11ths and 7ths.

2. Use your calculator to find the smallest number that is divisible by 2, 3, 4, 5, 6, 7, 8 and 9.

3. Given the difference of two squares: $x^2 - y^2 = n$, find what numbers n cannot be. Generalize your answer.

Examples for Discovering Relationships

1. Tin Can Mathematics

Use #5 tin cans. With a metric tape find the circumference and diameter of each tin can. Record these measurements in the chart below. For each tin can divide the circumference by the diameter and record the quotient in the chart. Express each quotient to the nearest thousandth.

Tin Can	Circumference	Diameter	C + D
1			
2			
3			
4			
5			

Find the average of the quotients that you calculated. Express this answer to the nearest thousandth.

What relationship did you discover between the circumferences and diameters of your tin cans?

Check with two friends about their findings.

2. Right Triangles

Using a centimetre ruler and centimetre dot paper, draw four right triangles and measure each side to the nearest millimeter. Record these measurements in the chart. (One is done for you in the chart below.)

Measure of sides

Triangle	a	b	c	a^2	b^2	c^2	$a^2 + b^2$
1	3.0	4.0	5.0	9	16	25	25
2							
3							
4							

Using your calculator, complete the chart.

What relationship have you discovered about the sides of a right triangle?

Examples for Developing Estimation Skills

1. What is the largest number that can be written given some fixed digits such as three 9's or four 2's? [For instance, $2^2)^2)^2$ or 22^{22}.] Check your estimate with your calculator.

2. What is the largest product of a two-digit and a three-digit number using the digits 3, 4, 5, 7, 8? What were the two factors? Try again using 1, 2, 4, 7, 9. Can you generalize your findings?

3. Indicate on the number lines below where each of the answers belong.

A

B

C

8 × 9 = ____________   1.2 × 2.4 = ____________
0.2 × 3.5 = ____________   4 + 2.8 + 0.05 = ____________

84.5 + 0.5 = ____________   0.17 + 0.003 + 0.641 = ____________

Reinforcing Inverse Relationships and Properties of Operations

1. Find the missing number in each problem using your calculator.

236 × ________ = 14868   What operation did you use to find the missing number? ____________

24973 + ________ = 53741   ____________

________ + 235.6 = 4.8   ____________

7.2 × ________ = 6.696   ____________

What relationships did you discover?

2. Complete the chart below.

a	b	a × b	a + b
65		1495	
	18		41
		96	20

Make up three more problems and give to a friend to complete.

3. Find the missing digits. Guess and check!

a.
```
  13_
 +2_
  38
```

b.
```
   378
  ×  5
  1890
 3024
```

c.
```
       3_
57) 4731
    -
    ____
     171
     171
```

4. Find the total value for N. Do you see a shorter way to do the computation?

a. $(2 \times 3) + (2 \times 4) = N$

b. $(37 \times 30) + (37 \times 1) = N$

c. $(1024 \times 53) + (1024 \times 34) = N$

d. $(8 \times 5) + (8 \times 6) + (8 \times 8) = N$

5. Supply the answers by inspection and check with your calculator.

a. $(3 \times 5) + 5 =$ __________

b. $(2149 \times 528) + 528 =$ __________

c. $(111 + 37) \times 37 =$ __________

d. $(5332114 + 1234) \times 1234 =$ __________

6. The opportunity to *play* with the calculator might whet the curiosity or stir the imagination of a student, especially if provided with one of the several paperback booklets on using the hand calculator. Many of these booklets have sections on number curiosities, puzzles and games.

7. For the algebra student

Of course the calculator cannot do algebra (a limitation that actually enhances its value in mathematics learning). But because students can quickly evaluate an algebraic expression for most real replacements for the variables with the calculator they can easily determine the correctness of their results for procedures such as the following.

a. Multiply $2x$ times $3x$.
b. Multiply x^2 times x^3.
c. Simplify $(x^2)^3$.
d. Multiply $(2x + 3)(3x^2 - 5x + 7)$.
e. Simplify $(4x^2 - 18)(8x + 16)$.
f. Solve equations of all types.
g. Solve inequalities of all types (by verifying the endpoints of the solution sets).

The Student as Instructor to the Calculator

The strategy of students instructing the calculator is appropriate for all students in all courses. For those students who have not mastered paper and pencil calculation, the calculator opens up the world of mathematics that perhaps has heretofore been closed. With the calculator, paper and pencil calculation is no longer necessary in order to use mathematics. Lack of skill in paper and pencil calculation need not and should not bar students from learning to use mathematics. Many students who could not do paper and pencil calculation reliably have shown remarkable insight into mathematical relationships and a surprising ability to see through word problems. These students may use the calculators to help them memorize the addition and multiplication tables and to do the same mathematics as the students who have mastered paper and pencil calculation.

1. Applied Mathematics and Consumer Mathematics

With the calculator on hand, put the emphasis where it belongs in the course — on problem solving rather than computational skills. Students will not be able to buy decision making machines, but they can buy inexpensive calculating machines. Here the goal for students will be to determine what to tell the calculator to do. Lack of calculating ability need no longer be a barrier to that goal. Real life problems now can be investigated in class. The problems no longer need to be carefully contrived and overly simplified to keep the length and tedium of the calculations within bounds. The textbook problems can still be used, and the students can concentrate on learning *what to do* to solve problems such as the following.

a. What does it really cost per mile to operate a car?

b. Can I save enough on financing costs while buying a car to make it worth while to shop around for the financing?

c. How many miles/kilometers does my car get to the gallon/liter?

d. How much is the APR on this easy term purchase?

e. How does the cost of renting compare with that of owning a house?

f. What information do I need?

g. Which is a better investment, savings certificates or common stocks or bonds?

h. How much interest is paid on a 30-year mortgage at 9% above that paid on a similar mortgage at 8½%?

i. How many cars would be involved in a one-lane traffic jam from New York to Los Angeles if the average length of a car is approximately 5.5 meters and the distance from New York to Los Angeles is 4,972 kilometers?

j. Simple and compound interest

P(1 + rt) represents the amount that must be paid when a principle (P) is borrowed at a simple annual interest rate (r) for t years.

Evaluate P(1 + rt) when P = \$6000, r = 9% and t = 3 years.

If \$750 is borrowed at 10½% interest rate for 15 months how much must be repaid?

Mr. M borrowed \$500 at 12% simple interest for one year. At the end of the year he decided to borrow for another year the amount he owed. How much did he have to pay back at the end of the second year?

HINT: first year 500(1 + .12)1 = 560

second year 560(1 + .12) = 627.20

or

$500(1.12)(1.12) = 500(1 + .12)^2$

Write the general formula for computing compound interest for borrowing P dollars at an annual rate r at the end of t years.

Mrs. K borrowed \$600 for 5 years at a compound interest rate of 15% per year. How much did she owe at the end of the 5 years.

k. Average

Measure and record the height (in centimeters) and weight (in kilograms) of twenty-five students in your grade.

Male/Female	Height	Weight

Compute the average height and weight of the males and females.

Make a scatter diagram showing the relationships of height and weight of males and females. Determine if there is a correlation.

Separate the height statistics for the males and the females into three categories — short, average, tall.

Separate the weight statistics for the males and females into three categories — light, average, heavy.

From the results of this sample, predict the number of students in your grade that would fit each category.

2. Algebra, Geometry, Trigonometry and Calculus

For the serious high school mathematics student, no other available tool matches the *programmable calculator* as an instrument for learning mathematical concepts and learning techniques of mathematical manipulation, with respect to economy, capability, ready availability to students and ease of use. At a relatively small cost, an entire class can be supplied with programmable calculators which will do all the mathematics the students will want done. Every student can have a hands-on experience whenever needed so that the calculator is genuinely a means to learning for the students. Writing programs for the calculator requires students to think through every step of the process, encourage imagination and ingenuity to shape the procedures so that they fall within the abilities and capacities of the calculator. As an added bonus students learn the principles of programming.

Furthermore, the programmable calculator — and to a lesser extent, the nonprogrammable scientific calculators — makes possible investigations of topics that are impractical without calculators. For instance

a. polar equations and their graphs (beyond a bare introduction),

b. parametric equations,

c. transcendental functions and their graphs,

d. rotations of axes,

e. area under a curve (in algebra classes),

f. "seeing" the converging of a series,

g. derivation and use of series for evaluating transcendental functions.

The programmable calculator opens up a new world of mathematics projects and makes possible the investigation of fascinating questions such as these.

a. What happens to the curve when the exponents in an equation of a circle or an ellipse are permitted to take on values all the way from 0 to 3?

b. Are there two parabolas that intersect at exactly three points?

c. What formula could be used to determine how much money would have to be placed into a savings account each p[illegible] at r annual interest rate, compounded K times a year in order to have A nu[illegible] dollars in the account at the end of n years?

d. Is there an easier and more efficient way to evaluate a polynomial than doing it term by term?

In using the programmable calculator, students should first be taught a concept and the techniques for using the concept. The students should use the techniques "by hand" in a few exercises. Following that the teacher should make the assignment of teaching the calculator to carry through the techniques by writing a program for it to follow. The students should check, debug, then make a tape of the program, and finally use it to solve a set of problems. The students' understanding will be enhanced since they cannot write a program without a thorough understanding of the mathematics nor will the calculator supply any missing steps or take anything for granted, and their skills will be more in keeping with the demands of the day.

The procedures above can be used effectively with topics such as the following.

a. Prime factorization of any positive integer

b. GCD and LCM of two or more integers

c. Evaluating a polynomial of any degree for a rational x

d. Finding solutions to a system of two linear equations in two variables

e. Solving quadratic equations for both real and complex solutions

f. Summing arithmetic series

g. Summing geometric series

h. Finding permutations and combinations of n elements

i. Coefficients for binomial expansions to rational powers

j. Summing up the first n terms of a binomial series

k. Writing the trigonometric functions in terms of the sine function

l. Solving triangles and finding areas of triangles given various sets of data

m. Converting complex numbers from rectangular to polar form and finding powers and roots of complex numbers

n. Finding approximations to real zeros of various functions

o. Finding approximate area under the curve $y = 1/x$

p. Evaluating a 3×3 determinant

q. Finding the return on an investment at various rates and periods, and finding the investment — either a single investment or periodic investment — necessary to yield a stated amount

To develop good habits of program writing, the students should write out each program completely and correctly. Each student may be given a program tape and use that one tape for every program. The teacher can easily check that the program works by running the tape through the calculator and solving a problem with the program. The students should be allowed to solve several problems using the programs they have written. Often problems that are less contrived will produce simpler solutions than those in the textbook and can be used for greater motivation.

If programmable calculators are not available but calculators with the transcendental functions are available, then everything except the recording of programs can still be carried out. *Programs* consisting of a sequence of keystrokes that must be used with the calculator can be written and then used. Time will probably not permit an extensive use of the programs as would be possible with the tapes.

Strategies for Using the Computer in the Mathematics Classroom

Instructional uses of the computer can be separated into two categories, those that require student knowledge of computer programming and those that do not. In the second category is computer-managed instruction including various forms of computer-assisted instruction, such as drill, tutorial, gaming and simulation, along with the use of canned programs to do the computations involved in problem solving. The computer is an essential tool for many people who apply mathematics to real-world situations.

Computer Applications

The computer should serve as an instructional aid for attaining existing goals and objectives upon which a modern mathematics program is built. The computer can be used to develop the topics that are normally stressed in the secondary school program. Computers can be used to study a variety of topics in mathematics such as

functions	trigonometry	vectors
hopper function	limits of sequences	law of sines and cosines
triangular condition	partial sums of series	calculus (including upper and lower Riemann sums
equation of a line	area under the curve	midpoint rule
inverse of a function	Pascal's triangle	trapezoidal rule
approximation of roots	approximation of pi	cones
sum	absolute value	numerical and Monte Carlo methods
difference	distance	geometric and arithmetic means
product	repeating decimals	standard deviation
quotient of complex numbers	quadratic formula	place value
matrices	proportions	
solving equations	variance	
probability and statistics using the mean		

Surely the computer is having a marked impact on education in the mathematics classroom.

Additionally, the computer has vast application outside the realm of the mathematics classroom. Computer programs can be used in setting up the calculations needed in chemistry, physics and other science courses. Computer music, computer dance, computer animated movies and computer-generated pictures are all interesting topics for a secondary school arts department to pursue.

Computers and related technology can be used in the schools with regard to the following activities.

1. Present unfamiliar concepts and principles in an interactive textbook format to students at computer terminals

2. Generate, administer, score tests and report results

3. Provide individualized tutoring to students

4. Drill students on new skills

5. Diagnose learning problems and error patterns

6. Help students learn algorithms for solving problems

7. Promote learning through discovery

8. Manage complex, multimedia learning environments

9. Provide students an additional measure of motivation to learn mathematics

There are many variables that govern the use of the computer in the classroom, such as the topic to be studied, the availability of computer facilities and the programming ability of the students. Each of these variables may have a broad range of values. There may be occasional or frequent access to a computer. Students may work in groups or individually. Students may be able to run canned programs, make minor modifications in existing programs or write their own programs. Each of these conditions affect and are affected by the planned use of the computer.

Program Types

The type of program used is the key factor in determining the success of a strategy. There are six major types of programs — drill, exploratory, branching drill, game, simulating and student-written programs.

Drill programs are canned programs that give students practice in areas such as arithmetic of whole numbers, multiplying binomials or finding roots of quadratic polynomials. The program may summarize the student's progress and store it in a file or print it at the terminal.

Exploratory programs are canned programs that perform extensive calculations for the student. These programs may be modified to suit the needs or desires of a particular student by supplying data or changing a particular line of the program (such as the definition of a function). Examples include tabulating functional values for the purpose of graphing, calculating the nth term of a sequence, calculating Reimann sums or calculating the means of several samples from a population (which may be used to construct a histogram to illustrate the distribution of the means).

Branching drill programs are drill programs that have the ability to determine the probable cause of errors made by students and provide substantive remarks toward correcting these errors. Sometimes problems related to the particular student difficulty are chosen with increased frequency by the program. (For instance, in a multiplication drill, if students have trouble multiplying by 7, such problems are chosen more often). Examples include a signed arithmetic drill that checks for proper calculation of a common denominator. A program on word problems could check where appropriate formulas are chosen and proper values are assigned to the variables.

Game programs provide students with amusement as well as motivation for applying mathematical concepts, learning new concepts and creating and writing programs for their own games. Few schools can provide computerized games solely for amusement purposes. However, a student who writes a program for a game may develop many programming skills. Games can provide more than amusement and these are most appropriate. Drill programs can easily be turned into games by adding a reward system. Other games encourage students to use their knowledge of probability to devise strategies for playing games.

Simulation programs model some physical process or experiment and produce observations of that process. For instance, students asked to flip a coin one thousand times and record the results are almost certain to become bored and not complete the task. However, a computer program could quickly simulate the process. Such programs may be canned or written by the students. Examples include coin flipping, genetics problems, cereal box prize problems and grocery store games. Simulation programs are particularly suited to illustrating ideas from probability and statistics.

Student written programs may provide students with opportunities to learn a great deal about topics when they write programs concerned with some aspect of the topic. The learning occurs since one must have a precise understanding and an appreciation for details of a topic before it can be programmed. The broadest range of mathematical topics may be covered by this type of program.

Resources and Topics

Assuming that the basic objective is to use the computer to enhance mathematics learning at several grade levels (and thereby attain indirectly many of the goals of computer-literacy or computer-programming classes), then where might one begin?

The most common approach is to give the students the task of writing simple computer programs which address some topic in the mathematics course. (For well over a decade problem solving with the computer has been underdeveloped by educators). There is a lot of good material in existence. However, two publications are so important that they are designated as required reading on the subject — back issues of *The Mathematics Teacher* and *Computer Assisted Mathematics Program* (CAMP). The serious mathematics educator is undoubtedly aware of the importance of *The Mathematics Teacher* but may not be aware of the excellent and current coverage which this publication has given for many years to mathematics topics that can be enhanced through the use of a computer (or calculator). The CAMP series may be less well-known since it is nearly 10 years old and (though most volumes are still for sale) is out of print. But the CAMP series was so thoroughly done that its age, even in the fast-moving computer field, cannot fault its quality as a curriculum reference for mathematics educators. It provided a model and impetus (and many of the topics) for the supplemental computer exercises which appear in many textbook series today. CAMP gives computer exercises which enhance such topics as Grade 8 - fractions, decimals and percents; squares and square roots; properties of the system of rational numbers; equations and inequalities; Grade 9 - absolute value; conditions with absolute value; distance between points in the plane; reducing terms of fractions; linear equations; Grade 10 - parallelism; quadrilaterals; area; Pythagorean theorem; similarity; coordinate systems and distance; Grade 11 - relations and functions; linear programming; the quadratic formula; matrices; Grade 12 - topics in calculus and analytic geometry; topics in the elementary functions and topics in probability and statistics.

Although many schools use A Programming Language (APL) the most widely used language is Beginner's All-purpose Symbolic Instruction Code (BASIC). Today's versions of the BASIC language can be quickly learned by students who have never programmed, yet, for the advanced student, provide many of the features of their more complex counterparts FORTRAN and COBOL.

The Mechanics of Implementing Computer Topics

Ideally, tere would be a room containing 20 or 30 computer terminals of varying capabilities connected to a time-sharing computer, monitored full-time by a computer-wise paraprofessional to help students complete their programming assignments. Dream on! Terminals are not yet that inexpensive, and, in most schools, integration of computer use into the curriculum has not come far enough for that facility to be cost-effective. What is happening is this. A school will have access to computer terminals which are capable of communicating either with a large time-sharing computer or with an in-house microcomputer system. The goal is to get the maximum number of hours of student connect-time from these limited facilities. Here are a few pointers toward that end.

1. Schedule time carefully. Sign-up sheet on a clipboard should be attached to the terminal. (Note: Most terminals do not require constant monitoring by faculty.)

2. Place the terminal in a room which has direct access from outside the building so that selected students can be given access at night. (Note: They could also process paper tapes prepared by classmates on an off-line terminal during the day.)

3. When possible, use portable terminals which can be checked out to students for use over phone lines at night. (Note: Most time-sharing arrangements offer less expensive computer time at night).

4. Use the batch method. There are machines available which allow students to write programs by marking mark-sense cards. Though this detracts from the students' experience with using an interactive computer terminal, it is an excellent way to permit a large volume of students to write programs and get computer-printed results for reinforcement.

But What If There Is No Computer?

How does the educator meet the needs and exploit the opportunities of computer applications if there is simply no computer available? Consider the following.

1. Beg, borrow or otherwise arrange. There are virtually no schools so isolated that no computer is nearby. Local businesses, banks and universities will frequently permit access to their computers for little or no cost, as long as such use does not interfere with their operations. Such arrangements may mean evening use at the company's site (student volunteers will readily accept such tasks) or perhaps a time-sharing line to the school. Computer vendors looking for a possible sale a few years hence may also be helpful. PTA or other community groups have also helped fund such operations. Basically, it is wise to remember that availability of computer facilities is more often a function of the educator's enthusiasm than physical limitations.

2. Computer courses and topics are sometimes taught without computer access. Most tests and outlines of mathematics topics are largely computer-independent, and, although the hands-on experience is undeniably a valuable motivational factor for students, a certain amount of enthusiasm is usually retained without the machine, merely because of the different approach to the "same old subject."

3. Field trips to computer installations, movies about computer applications and guest speakers (parents and local professionals) can enhance a section of computer-related mathematics topics. Engineering firms which use computer graphics and banks which have large data-processing volume make excellent field-trip sites.

Conclusion

Inclusion of computer-enhanced mathematics topics in the curriculum permits operating optimization of instructional-computer use for the largest number of students, and help mathematics educators respond to the growing need for computer literacy. There are topics at all grade levels which are well-suited for teaching with computer aid, and there are sufficient publications available to provide educators with tried and proved materials. The challenge, then, to the mathematics educator is to carefully research the literature, consult with colleagues already using computers and integrate desirable topics into the mathematics curriculum.

Strategies for Working with Students through Independent Study

There are many misconceptions about the nature of independent study. Often it is thought of as unstructured and unevaluated work, designed to provide more freedom for the teacher. Some view it as selfish and antisocial on the art of the student, while others believe that a simple change in the rate of study constitutes independent study.

Independent study does not just happen. A great deal of planning and organization are required of both the teacher and the student. In order to be effective, independent study must be structured. This paper will provide a number of techniques for structuring independent study.

Identifying Students Who will Benefit from Independent Study

Independent study is a learning mode that does not characterize a particular group of students but describes a way in which an individual student may most effectively learn a specific topic. Because gifted students learn quickly and easily, they are often considered more successful at independent study than other students. Many have been learning independently for years. Independent study provides a viable alternative for gifted students who are not being challenged in the regular classroom. However, if the gifted student is not genuinely interested in the topic of study, then the student will not put forth the necessary effort and will learn little.

A student, of course, need not be gifted in order to benefit from independent study. The most successful independent study students may be those who are creative and self-motivated. A student who is especially interested in learning about a particular subject will work diligently at it and will require little prodding from the teacher.

Interviewing is a valuable technique for determining whether a student will be successful at independent study. The teacher should ask the student what work has been done. The response need not relate to mathematics nor need the response necessarily be school-related. If the student has ever made a collection such as stamps or coins, entered a project in a science fair or studied some aspect of history such as naval battles or the civil war, then the student may be a good prospect for independent study. The teacher should be alert for evidence that the student has spent a portion of free time working on a project reflecting the student's interest.

Helping the Student Get Started

The teacher should begin by sharing with the student the philosophy of the program, describing any administrative limitations and explaining exactly what will be expected. Next, the teacher should work with the student to determine the topic to be studied. Often it is helpful to provide a list of alternatives. Unless the student already has a definite topic in mind, frustration may result if faced with unlimited choices.

After the topic has been determined, the teacher must help the student locate resource material. It is generally better to show the student how to find material rather than for the teacher to actually gather the material. Research skills must be taught to those students who have not yet acquired them. As the student locates materials which may be useful, the student should list them in the form of a bibliography.

Planning and Organization

After completing the bibliography, the student should make a contract or plan which includes a statement of goals and objectives and the activities that will be performed to achieve these goals and objectives. It is a good idea to have the student set deadlines for completion of each activity. This will require the student to budget time and it will provide a means for the teacher to determine whether or not the student is on schedule. The contract should be signed by the teacher, the student and the student's parents.

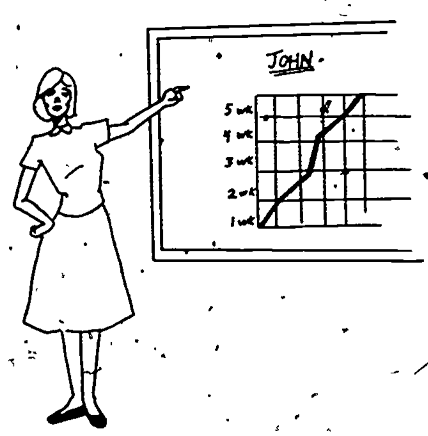
Students sometimes object to writing a contract on the grounds that if they knew enough to do that, they would not need to study the subject. At this point, the teacher should stress the importance of planning and explain that the first task is to find out enough about the subject so that the student may plan effectively.

Another technique for good planning is to have the student submit a weekly statement outlining plans for the coming week and describing the progress made in the past week. This may be compared with the contract to determine if the student is working on schedule or if the schedule should be revised.

Formative evaluation — and subsequent revision — should be applied to the plan for the study in terms of appropriateness of the various components such as goals, objectives, activities and schedule.

Requiring A Product

All independent study students should be required to produce some tangible product as evidence of their learning. In other words, they should have something to show for all the work they have done. Products which receive recognition from others are especially rewarding. Students should be encouraged to enter projects in mathematics fairs, present papers at student symposiums, submit papers to mathematical journals or present lectures to mathematics clubs or mathematics classes.



Evaluating Student Progress

One of the most important aspects of evaluation is self-evaluation. The student should frequently be asked to state, either orally or in writing, what progress has been made toward reaching the goals and objectives. Thus, the contract will serve as the basis for evaluation.

The teacher must evaluate the quality of the product, keeping in mind the ability of the student. Signs of increased self-discipline and self-direction may also serve as evidence of progress.

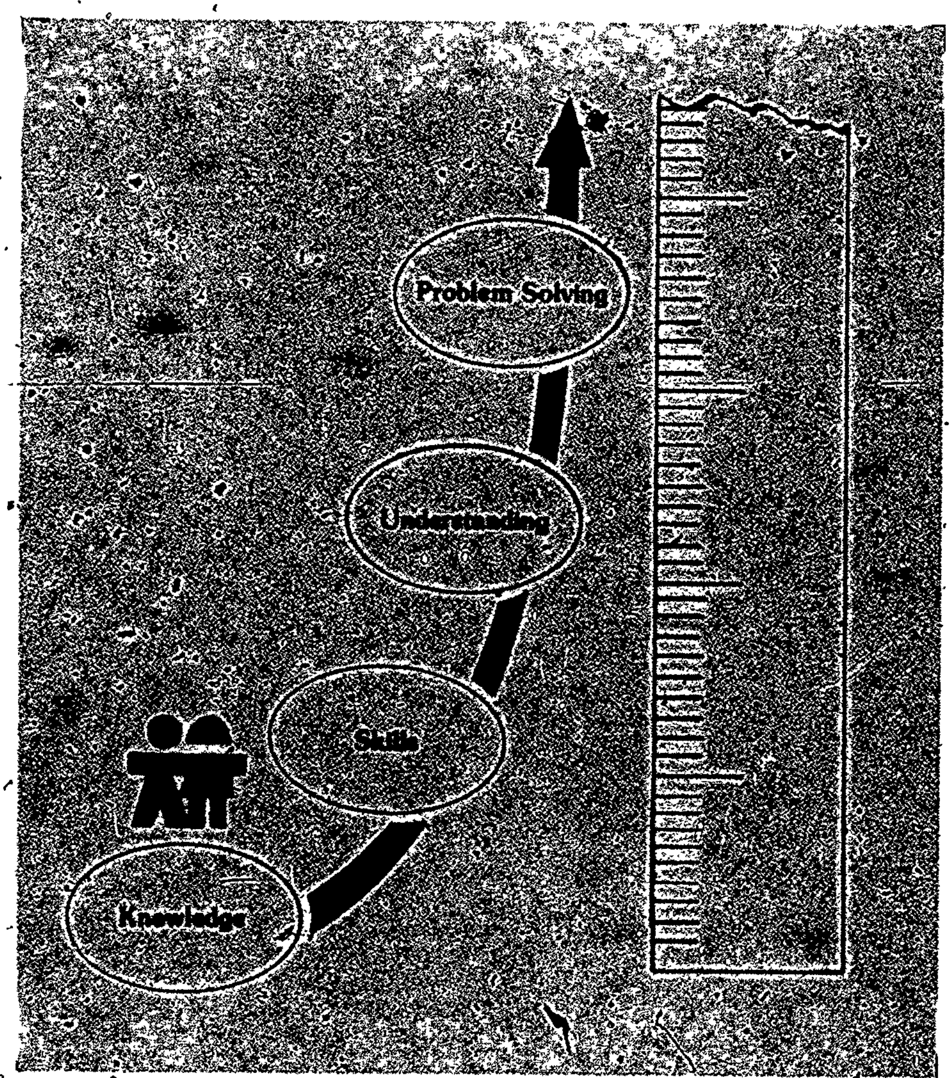
Conclusion

Independent study is an extremely valuable learning method for some students. The teacher should not limit a student's range of studies to those subjects which the teacher knows best. If the student progresses beyond the level of the teacher, then the two simply exchange roles. This can be highly rewarding for both.

Suggested Topics For Independent Study

The following list is intended to give a general idea of the kinds of topics that are suitable for independent study. It is by no means complete.

Number theory	History of mathematics
Sequences	Problem solving
Series	Unsolved problems
Modular arithmetic	Topology
Numeration systems	Infinity
Gaussian integers	Pascal's triangle
Prime numbers	Decision theory
Finite arithmetic	Information theory
Group theory	Linear programming
Game theory	Probability
Graph theory	Symbolic logic
Projective geometry	Boolean algebra
Non-Euclidean geometry	Switching circuits
Solid geometry	Mathematical models
Braid theory	Computer programming



EVALUATING

Evaluating Mathematics Learning

In several other sections of this guide there are various suggestions for evaluating mathematics learning. The purpose of this section is to provide a point of view on evaluation, develop a framework for discussing evaluation and relate this material to the rest of the guide. A fundamental assumption is that carrying out evaluation of mathematics learning requires expertise in mathematics teaching. It is not a task that can be left to the outside specialist. The evaluation of mathematics learning must follow from our objectives and strategies for mathematics instruction. The process must not be allowed to reverse itself and let the principles of evaluation dictate the objectives of mathematics instruction.

Why Evaluate?

Information about mathematics learning is usually gathered because a decision of some sort has to be made. Has the student learned a specific objective? Have objectives been learned to a satisfactory level? Should a particular textbook be used? Has a program worked? Evaluation of mathematics learning, therefore, is concerned with input into the whole range of decisions that must be made in organizing and implementing a mathematics program.

Evaluation is an integral part of the instructional process. Assessing the mathematics performance helps students set personal goals, provides them indications of their progress and can develop motivation. It is a source of feedback and reinforcement. It can provide a sense of stability and organization for the mathematics course as perceived by the student.

Evaluation also serves to provide information to the various groups to whom the schools should be accountable. This includes parents, the general public and those who determine school policies and funding. Mathematics evaluation provides not only the opportunity to describe how programs are doing, through test scores and the like, but also, if thoroughly reported, the chance to discuss the nature of the program. Unfortunately, inadequate reporting often provides statements of test scores, no discussion of the program and a misrepresentation of the mathematics program by the implication that the content of the test covers the curriculum.

Mathematics evaluation is also done from necessity. It is prescribed, in some cases, by mandates from outside the school system.

Given that several answers to the question "Why evaluate?" can be given, it is clear that mathematics evaluation will be done and it should be done well. To do it well requires the interest, involvement and direction of mathematics teachers at every step — from selecting objectives to reporting and using the data.

What is Evaluation?

A lot of evaluation activities are part of operating mathematics programs. These activities include the assessing of mathematics learning, the selection and improvement of instructional materials, the observation of teaching techniques and the judgment on the quality of mathematics programs. The concern in this statement is with evaluating mathematics learning and mathematics programs, but evaluating is much more than testing. What, then, is evaluation?

To be sure, valid mathematics tests are important for some of the evaluating of mathematics learning and programs. But, they are instruments of the evaluation, always with certain limitations, and the data from tests are only one element of the evaluation.

An evaluation is analogous to telling a story. The evaluation must weave together all of the information, of which test data may be an important accumulation of facts. Other information may be critical to the story, e.g., observations of the students, observations of the testing, descriptions of the program, the teaching or the students' background. The evaluation process finally, at some point, rests on judgments. The human mind is also an evaluation instrument. Judgments are made in the selection of tests, in the interpretation of the data and in making decisions based on the evaluation.

The evaluation process, therefore, is a sequence of information gathering, interpretation and decision making. In determining a student's grade for a unit, the teacher gathers information from tests, homework, classroom interaction and observations; the teacher makes judgments on the relative importance of the different kinds of information; and the teacher arrives at a grade. The teacher is surely guided by concerns for quality of the mathematics, fairness, validity and reliability of information and consistency, but the sequence of information gathering, interpretation and decision making has been followed. The letter grade is only a small part of the whole picture. Likewise, in evaluating a mathematics program, someone gathers a lot of information (e.g., description of objectives, data from tests, descriptions of the program or observations), provides an analysis of the information and presents the result in a manner reflecting an interpretation of what it means.

What is Evaluated?

When mathematics evaluation is mentioned, many aspects of mathematics programs, teaching and learning may be the object of inquiry. The following is a partial list.

Evaluating students' learning of

— mathematics knowledge
— mathematics skills
— mathematics understanding
— mathematics problem solving
— attitudes toward mathematics
— appreciation of mathematics
— mental skills in mathematics

Evaluating mathematics programs

— goals
— students' performance
— judgments

Evaluating mathematics teaching

— observation
— self-evaluation
— microteaching
— the effective teacher

Evaluating mathematics materials

— textbooks
— tests
— supplementary materials
— nonprint materials
— computers
— calculators
— teacher-made materials

As stated earlier, the concern here will be with the evaluation of learning and the evaluation of programs. To be sure the evaluation of teaching and the evaluation of materials (selection/purchase, instructional; how well they work) are extremely important evaluation activities. The evaluation sequence is similar, but the techniques required are quite different.

A Framework for Thinking About Mathematics Evaluation

A useful technique to discuss particular evaluation tasks in mathematics is to construct a framework or model. There are many examples in the literature, most of which have some connection with the early work of Bloom (1956). One of the most comprehensive is the chapter on the evaluation of student learning in mathematics by Wilson (1971) in which there is a detailed development of a model followed by extensive examples. Other examples are provided by Begle and Wilson (1970), the National Assessment of Educational Progress (1978) and by Avital and Shettleworth (1968).

The basic strategy of these models or frameworks is to discuss mathematics objectives, mathematics test items or other types of mathematics performance, e.g., observations in terms of two dimensions — mathematics content and cognitive processes used in doing mathematics. Both the content strands and the processes categories have been stressed elsewhere in this guide. They are summarized here.

- Content Strands

  Sets, Numbers and Numeration
  Relations and Functions
  Operations, their Properties and Number Theory
  Geometry
  Algebra
  Probability and Statistics
  Measurement and Estimation
  Computing and Computers
  Mathematical Reasoning and Logic

- Processes

  Mathematical knowledge
  Mathematical skill
  Mathematical understanding
  Mathematical problem solving

PROCESSES

CONTENT	Mathematical Knowledge	Mathematical Skills	Mathematical Understanding	Mathematical Problem Solving
Sets, Numbers and Numeration				
Relations and Functions				
Operations, Their Properties and Number Theory				
Geometry				
Algebra				
Probability and Statistics				
Measurement and Estimation				
Computing and Computers				
Mathematical Reasoning and Logic				

A framework for specifying mathematics learning objectives and outcomes.

The two-dimensional grid on the preceding page provides four cells for each content strand, emphasizing that for each strand there should be a concern for knowledge, skill, understanding and problem solving. It is assumed that the process categories tend to be hierarchical from knowledge to problem solving! An objective or test item that deals with problem solving will also require certain knowledge, skills and understanding. An understanding item may require the use of subordinate skills and knowledge.

Constructing and using such a grid should concentrate on the speed of processes for each strand rather than where the dividing line between processes might be. The grid is merely a framework, a heuristic device. If there seems to be doubt as to whether an item or objective measures understanding or problem solving, use it in either. The important point is to recognize that there is a gradation in the complexity of objectives, tasks or test items for each strand, from recall of facts, to skills, to understanding, to routine problem solving, all the way to creative problem solving.

- **Examples of using the framework**

The objectives in Collection C, those mathematics objectives presumed necessary for productive citizenship, have been placed (by number) in the grid on the following page. Some objectives fit in more than one strand; some fit in more than one process column. Readers may not agree with either placement, but Collection P and Collection E could be examined in the same way (an exercise for the reader).

CONTENT	PROCESSES: Mathematical Knowledge	Mathematical Skills	Mathematical Understanding	Mathematical Problem Solving
Sets, Numbers and Numeration	4, 7, 42, 43	1, 2, 3, 7, 28, 30, 32, 40	30, 37, 38	34, 38, 41
Relations and Functions	4, 24, 42, 43	1, 2, 3, 5, 14, 19, 20, 23, 28, 29, 30, 40	23, 30, 37, 38	19, 34, 38, 41
Operations, Their Properties and Number Theory	7, 8, 9, 42, 43	5, 7, 10, 11	12, 37, 38	34, 38
Geometry	15, 42, 43	15, 16, 17, 18, 19, 20, 21, 23	12, 23, 37, 38	19, 22, 34, 38
Algebra	42, 43		12, 37, 38	34, 38
Probability and Statistics	35, 42, 43	28, 29, 30, 31, 32, 33, 35, 36	12, 30, 37, 38	33, 34, 38
Measurement and Estimation	24, 25, 42, 43	19, 21, 23, 25, 26, 27, 29	12, 23, 37, 38	19, 22, 34, 38
Computing and Computers	4, 7, 8, 9, 13, 35, 42, 43	5, 6, 7, 10, 11, 19, 23, 29, 30, 35	12, 23, 30, 37, 38	19, 34, 38
Mathematical Reasoning and Logic	42, 43	39, 40	12, 37, 38	34, 38, 41

A framework applied to collection C objectives.

Some of the objectives are so broadly stated that more than one process level is included; e.g., some objectives are "and" statements where one part is skill or knowledge and the other is problem solving. A similar exercise on test items to measure these objectives will be more specific. Except for some very general objectives, however, it is clear that the bulk of Collection C deals with mathematical skills and the associated knowledge. A reservation must be attached to this observation in that some objectives (e.g., objective 6) deal with a much larger chunk of the curriculum than others (e.g., objective 35).

When the grid is used in program planning, writing tests or interpreting the evaluation, judgments must be made as to the relative emphasis assigned to each cell (that emphasis could be zero, but it ought to be a conscious choice). Each cell could, for some purposes, define a measure or, for other purposes, a strand or a process column could define a measure. Often a single test is purported to measure the whole grid when in fact only a few cells would have items from the test. More often the test user has not given adequate thought to what the test has measured.

An adaptation of this strategy for a particular unit or course would be to replace the general content strands with the particular topics in the unit or the course. Then the grid could be used to analyze the objectives of the unit or course, decide relative weights and emphases and to construct tests.

Program evaluation could use the grid in the discussion of results of testing. The cells in the grid could be used to highlight strengths and weaknesses. For example, National Assessment uses their grid to organize the reporting of results according to process levels (NAEP, 1978).

Another example, where a standardized test has been used, would be to classify the items from the standardized test into a grid and use this classification in the description of the results. Part of the discussion of results must deal with the congruence (or lack of it) of the standardized test with the goals of the program.

Techniques

Given the broader view of mathematics evaluation as information gathering, interpretation and decision making and the grid as a heuristic device for organizing the process, what various techniques could be used? There are many.

No technique can guarantee good quality. The task of the mathematics teacher is to apply good quality in any technique. Writing or asking good mathematics questions, free of error, in clear language is a requirement that transcends all techniques. The quality of the mathematics is not a function of any particular technique. On the other hand, certain techniques can be limiting, e.g., multiple choice questions. Mathematics teachers must monitor the quality of any technique.

- **Classroom Tests**

Making and interpreting tests is a major professional activity of mathematics teachers. The grid serves the purpose of helping enhance the content validity of the test — to help insure that the test measures what in fact has been taught and studied. Whether a topic is specific (one cell) or comprehensive (a whole strand) the correspondence of the items to the classroom activities can be addressed with the grid. Written tests are a major instructional tool.

- **Homework**

Most teachers use assignments, whether homework or seatwork, as an adjunct to instruction. One becomes proficient in mathematics by doing mathematics. Various management schemes can be used to incorporate students' performance on homework into an evaluation. For example, some teachers score each assignment, some score only selected assignments, some use student checking, some note only the completion or noncompletion. Regardless, the homework performance can be one source of evaluation information.

- **Projects**

Some courses lend themselves to independent study and extension. Assigning and grading a student project can be a means to assess this aspect of a students' mathematics. Several courses outlined in this guide suggest this evaluation technique as one part of the evaluation strategy. The project can lead to either a written or oral report.

- **Interview**

A dialogue with a student, as the student solves a problem, can provide considerable insight into the student's thinking. It can help to pinpoint strengths and weaknesses in the student's mathematics. It can provide the teacher information for instructional decisions.

- **Oral Quizzes**

Often a short oral quiz to a student can provide the teacher information to confirm whether the student is ready to move on to a new topic. It can give the teacher a chance to probe if a minor error has been committed: It can give the student a chance to explain.

- **Standardized Tests**

Standardized tests are of limited use because of the expense and because they seldom fit the course very well. The misuses of standardized tests are legend. Nevertheless, standardized tests do provide norms and therefore a benchmark against which performance can be gauged.

- **Criterion-Referenced Tests**

Criterion-referenced tests have items or sets of items associated with specific objectives. They are no more appropriate or valid than those specified objectives insofar as application to a specific program, school or class. Hence, if the objectives do not fit then the test will not fit either.

- **Student Performance Prescription**

Some individualized courses can make use of student prescriptions. There are several forms and management schemes for these, but essentially a contract is written for the student specifying the payoff — the grade for accomplishing particular assignments.

Evaluating Mathematics Affect

Attitudes toward mathematics, anxiety about mathematics and appreciation of mathematics are learned outcomes in mathematics programs. They are the concern of mathematics teachers. Feelings and beliefs are important. They influence students in subtle ways that are not well understood. Clearly, positive attitudes are desired, undue anxiety is not wanted and every student should appreciate mathematics.

A formal discussion of attitudes, anxiety or appreciation is not required here. The chapter by Wilson (1971) and the national assessment objectives booklet (NAEP, 1978) devote considerable space to these topics.

Informally, however, teachers should be alert to the affective goals of mathematics. Some aspects of these goals are the following.

- **Mathematics in Schools**

How do students feel about the mathematics they experience in school? Do they like it? Do they think it is important? Do they think it is easy? The national assessment has assessed these attitudes by asking questions about specific mathematics topics, such as solving word problems and working with fractions.

- **Mathematics as a Discipline**

What are students' beliefs about mathematics as a field of study? What are students' beliefs about mathematicians?

- **Mathematics and Society**

What are students' beliefs about the value of mathematics to society or to the individual in the society?

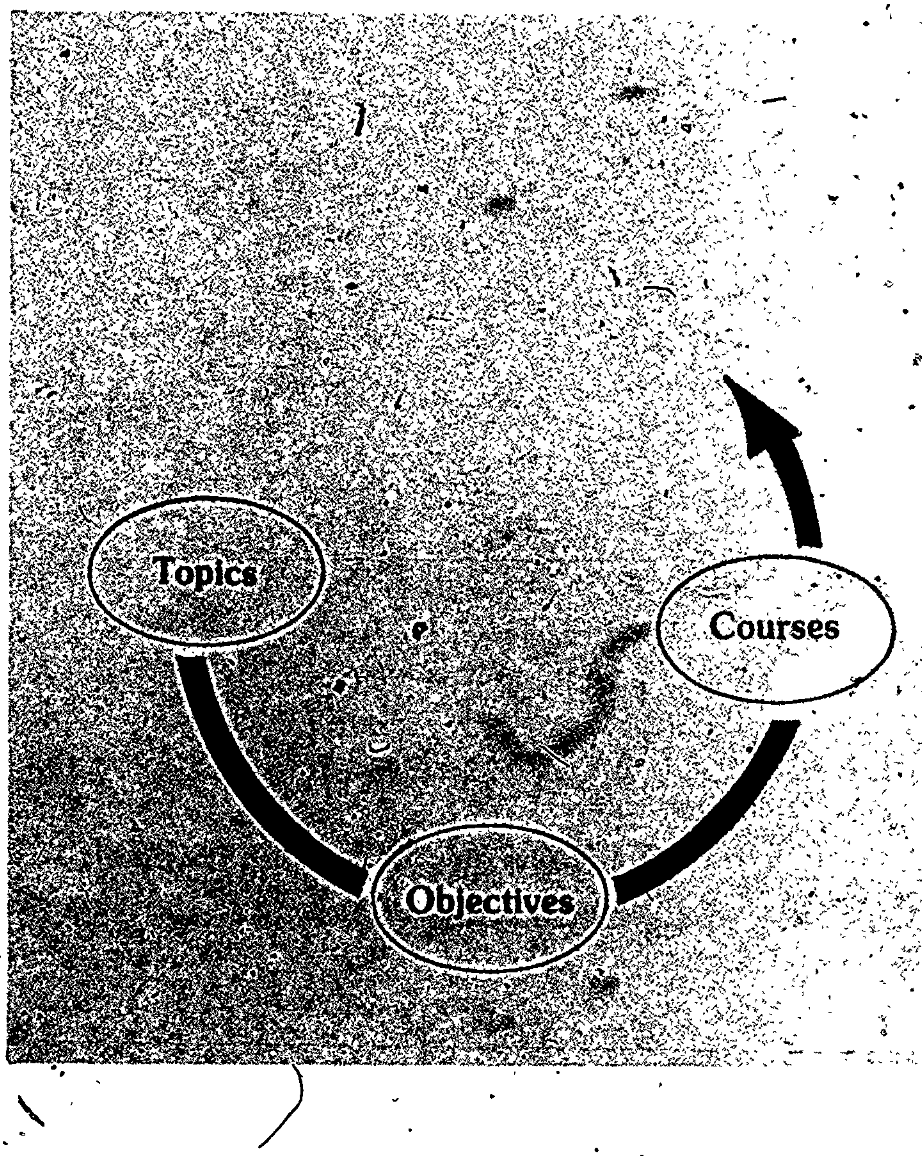
- **Mathematics and Self**

How do students internalize feelings about themselves in relation to mathematics?

These are four categories of mathematics affect assessed by national assessment (NAEP, 1978). The teacher can assess attitudes by observation, by talking with students, by listening to students or by judgments. Self-report instruments, such as attitude scales, are of limited value and are not easily interpreted. Mathematics appreciation can be assessed informally by observation and talking with students, but it can also be assessed by written assignments (and corresponding instruction) that deal with the value of a mathematics topic.

Summary and Challenge

Evaluation of mathematics programs and learning permeates much of what mathematics teachers do. Yet evaluation is a tool and an adjunct to higher quality mathematics programs. The goal is to have the best possible mathematics experience for each child. Evaluation can help accomplish that goal by providing information to the teacher for improving the experience. The challenge is that our evaluations must be of the same high quality as our instruction, yet truly serve that instruction rather than dominate it.


Topics
Courses
Objectives

Mathematics Content and Objectives — Point of View

In an age and society characterized by change, the secondary mathematics curriculum must be flexible enough and forward looking enough to accommodate students with diversified and changing career goals. This guide, *Mathematics for Georgia Secondary Schools*, is designed to help local curriculum committees and teachers plan content that is appropriate to the needs of their high school populations. See the section "Developing a Mathematics Secondary School Curriculum" for suggestions on steps for developing a curriculum.

The mathematics content of this guide is organized into strands or topics as the basis for the core of the curriculum. The objectives are grouped into three collections. Collection C contains those objectives which give the knowledge, skills and processes presumed to be necessary for productive citizenship. The objectives identified in Collection P are presumed prerequisite for postsecondary studies not requiring concentration in the field of mathematics. Collection E contains content which should prepare students for postsecondary studies requiring extensive mathematics.

These collections should in no way suggest a tracking of students. Practically, a student might change educational and career goals one or more times during his or her high school years and might opt to change a mathematics program of study accordingly. Emphasis should always be on flexibility of options.

The strands identified in this guide are not discrete. As evidenced by the code-keying of the objectives, many relate to two or more of the strands. The use of identifiable strands or topics merely provides convenient organizational threads. Furthermore, the objectives are not all-inclusive. The intent is to suggest the scope and depth of a given topic for a specified collection of students.

Problem solving should be the overarching goal of all mathematical study. For this reason it has not been identified as a strand but is presumed to permeate all strands. For it is only the application of mathematical knowledge, skills and processes to problems that gives validity to their inclusion in the curriculum.

Problem solving is not intended to be restricted to verbal, textbook-type problems. Teachers should seek realistic applications at all levels. In addition, problem-solving strategies should include not only the routine application of algorithmic-type procedures but also should include such strategies as the application of inductive and deductive methods; the use of pattern searching and generalization with verification by logical or analytical methods; the gathering, organizing and interpreting of information relevant to the solution of a particular problem.

The content of this guide is intended to serve as a framework for committees or teachers in designing a mathematics curriculum at the system or school level. The refinement and organization of the objectives into particular courses should be the function of the local committee. A sample of course descriptions is provided as an existence model but is not intended to be definitive. Ideally, each system will adapt the content of the state guide to the unique needs of its particular student population.

An extensive bibliography is provided in the guide. It is intended as reference for teachers and committees in their attempts to use and improve the guidelines included here.

Sets, Numbers, Numeration

With varying levels of sophistication in treatment, notions of sets and set relations permeate the mathematics of the high school curriculum. At the most elementary level, set ideas form the basis for the development of classification skills and logical language. Even if not explicitly stated, set concepts underlie the precise and correct use of logical terms such as some, all, none, at least and at most.

Matching sets in one-to-one correspondence gives meaning to the cardinal and ordinal numbers that are basic to everyday arithmetic literacy. Informal and intuitive use of set relations and language provides a basis for number relations as well as a convenient language for the discussion of geometric ideas as point sets.

Numeration provides the means for naming numbers that is essential to communicating number relations. At early levels of learning, an understanding of base-ten numeration is essential for effective use of the computational algorithms.

At the minimal level of competence for all students, an understanding of number is needed for meaningful application of numbers in everyday problems. Hence Collection C objectives include number and numeration concepts and skills. The order concepts of first, second, . . . last and the cardinal uses of as many as, more than, less than, form the basis for daily use of number. The notion of grouping by tens gives meaning to a place value system of numeration. Its application to naming rational numbers in decimal form is at the heart of numerical communications in a commercial-technological society.

In Collection P objectives, number concepts are extended to complex numbers with the primary emphasis on real numbers and their applications in problems. Numeration includes such extensions as naming rational numbers as repeating, nonterminating decimals; using exponential forms in scientific notation. The inclusion of number concepts such as betweenness and density provide the necessary conceptual background to enable further mathematics study if later career decisions demand it. It is important to keep doors open to changing and developing career opportunities for students who strive to attain Collection P objectives. It is better to give too much than too little.

For the E Collection, a formal treatment of sets and number is essential for the more sophisticated classification schemes necessary for the student of precalculus mathematics. This content includes a formal treatment of the concepts and language of elementary set theory including the construction and analysis of proofs. In addition, an abstract treatment of the concept of number system is essential, with particular attention to the complex numbers to provide the necessary background in this strand for extensive postsecondary study in mathematics.

Operations, Their Properties and Number Theory

Fundamental to using numbers in problem solving and in everyday application is an understanding of operations on whole numbers and on rational numbers. Fundamental to advanced study in mathematics is a sound knowledge of the real numbers as a system and the ultimate extension to complex numbers. The notion of a system resides in a set, operations on the elements of the set and properties of those operations.

Properties of the operations on whole numbers and rational numbers are basic to an intelligent use of the algorithms for computations and the applications of operations in problem solving. The very memorization of multiplication facts is made easier if the commutativity of multiplication is recognized. (If one knows 3×9, one also knows 9×3.) Similarly, it does little good to know the subtraction algorithm applied to base-ten numerals, if one is not aware of the noncommutative nature of subtraction. It is not the word *commutative* that is important at the minimal level of use, but it is the realization that $24 - 16$ is different from $16 - 24$; or for addition, that $24 + 16$ names the same number as $16 + 24$.

Thus, in the Collection C objectives, the properties of operations assume importance relative to their use in applying the operation or in translating the operation into computational techniques. For example, the multiplication algorithm makes sense as an application of the distributive property of multiplication over addition.

Number properties such as odd or even are translated into everyday applications, e.g., north and south interstate highways are odd-numbered, east and west highways are even-numbered; gas rationing schemes use odd-even numbered tags to designate days for gasoline purchases. In addition, a knowledge of divisibility, primes, least common multiple, greatest common divisor make computation with fractions meaningful.

Students aspiring to Collection P objectives should extend the ideas of this strand to the more abstract level of identifying and applying the properties of operations defined over a given set. The extension of number to the complex number system should be included in order to provide the basis for the solution of quadratic as well as linear equations. These students will need to extend number theory ideas to more advanced concepts such as the fundamental theorem of arithmetic and divisibility.

For students seeking to attain Collection E objectives, sets other than numbers with appropriate operations should lead to the identification and study of abstract algebraic structures such as groups, field and finite systems. Proof, in the context of such systems and including mathematical induction, should be an integral part of the secondary mathematics curriculum for those students who expect to continue with extensive postsecondary study of mathematics.

Relations

It is commonplace for mathematics teachers to say that relations are important in mathematics. However, this importance derives less from the general concept of relation than it does from the fact that the concepts of function, order and equivalence are subsumed under the concept of relation.

The notion of relation is easy to justify and to translate into objectives for Collections P and E where a formal treatment in the mathematical sense is appropriate. However, it is in the search for and identification of relations among sets of numbers, classification and seriation activities, graphing simple functions, that students attempting to attain Collection C objectives have an opportunity to see mathematics as more than mere computation and the rote application of meaningless rules.

Recent National Assessment of Educational Progress results indicate that the "back-to-basics" drive in the sense of the development of rote computational skills has been effective. Students at all levels tested showed improvement in computation with whole numbers. However, on questions that involved reasoning or applications, there was a significant drop in the percentage of correct responses.

Certainly, even in the most basic courses of the secondary curriculum, if problem-solving abilities are a desired outcome for all students, the ability to discern relations and to express them in a useful form is essential. Graphs, tables, simple formulas and equations are a part of every consumer's daily encounter in the media.

Students working toward Collection C objectives must, at minimum, be able to express and use simple relations in a formula form (i.e., area and perimeter) or in a tabular form (i.e., taxes and interest). In addition, not only should every consumer of graphic information be able to read and interpret tables and graphs (i.e., to extract information from a graph) but also should be able to make reasoned and intelligent extrapolations. Every consumer of statistical data should know when and whether the extension of a graph to show trends is reasonable and appropriate.

For all capable students the objectives keyed to the relations strand in Collection P should be achieved. This set of objectives to be achieved includes an extension of the relation concept to definition by rule, mapping or ordered pairs. It includes applying properties of equality and inequalities in a more formal way.

In addition, sets of ordered pairs of numbers which satisfy given conditions should be related to sets of points in the Cartesian coordinate plane. Graphing as well as other solution techniques should be applied to linear equations and inequalities.

For those students attempting to attain Collection E objectives, a comprehensive treatment of the relations concept should be provided. The order relation on the set of real numbers is so intrinsically important that the curriculum should provide students not only with the understanding of order as a relation but with the important consequences of order in the set of real numbers, including open and closed intervals, the image of intervals under functions, upper and lower bounds. The explicit study of equivalence including the concepts of reflexivity, symmetry and transitivity should be an integral part of the mathematics curriculum for students capable of Collection E objectives.

Geometry

Historically, geometry is a synonym for mathematics. It is undoubtedly the oldest organized deductive system, dating back to Euclid and the third century B.C. In spite of the introduction of proof into high school algebra in the 1960's, many teachers still consider geometry as the primary vehicle for introducing students to deductive methods. This belief has been substantiated by several teacher surveys.

In recent years, controversy has raged among mathematics educators over (1) the content of high school geometry and (2) the devotion of a full year's course to geometry. There are those who advocate the inclusion of geometry into an integrated program. Others prefer the maintenance of a year's course in geometry but with a transformational, vector or coordinate approach.

None of these controversies has been resolved, and it does not seem likely that there will be drastic changes in the academic content of high school geometry as stated in Collection E objectives in this classification. The objectives at this level reflect a strong commitment to the role of geometry in the development of an understanding of and appreciation of proof. There is a concession to the need for a broader, eclectic program in the extension of objectives to include transformational and coordinate proofs as well as an introduction to networks.

At other levels there is even less agreement nationally on what should constitute geometry content. Certainly, for all individuals, geometry should provide the opportunity to learn about space both in a relation context and in a metric context. Spatial perceptions and relations are a part of existence in a three dimensional world. To this end the objectives for Collection C are primarily concerned with relations between and among shapes that lead to identification and classification of figures. Simple transformations are suggested as basic to the development of spatial intuitions about objects and their images. Geometry at this level should be presented as a means of understanding and communicating spatial ideas, a means of "grasping space." Objectives should be achieved without the formality of deductive proof and should depend on generalizations derived from intuition and empirical verification.

Collection P provides an intermediate stage in the treatment of geometry. In addition to the minimal, intuitive geometric concerns for productive citizenship, these students should have more extensive experience with classifications and formulas. They should have some work with simple networks. They need coordinate geometry to enable the graphing of simple equations and inequalities. The treatment for these students may include some use of deduction in deriving relationships but at a less formal, rigorous level than that expected of Collection E students.

Algebra

High school algebra provides access to almost all of the other more advanced mathematics courses. Every student who is capable of achieving the abstract thinking required for success in elementary algebra should take at least one course at the high school level. Many career choices will be closed to the student who opts to terminate his or her mathematics without such a course.

Students who bypass high school algebra close the door to a reconsideration of career choices that require just a little more mathematics or they face the handicap of having to take algebra as a noncredit course in postsecondary study. For many students, delaying a first course in algebra until the tenth or eleventh grade may allow for the development of the maturity necessary for the discipline required in the more formal mathematics of an algebra course.

For students seeking to attain objectives in Collection P, one year of high school algebra should provide the minimal background for success in nonmathematical study or careers. This year of algebra should include the solution of linear equations and inequalities, graphing such functions and relations, transforming linear equations into slope-intercept form and relating this form to the graph.

As indicated in the objectives in this collection, an even better preparation would be supplied by two years of algebra which could extend the topics beyond simple algebraic relations and functions to quadratic functions. This study also would enable the relation of the algebra of matrices to transformations of the plane.

For students striving for Collection E objectives, high school algebra should be extended far enough to provide the theoretical basis for the traditional calculus-analysis directed curriculum still generally accepted as the appropriate mathematics content for entry into postsecondary studies requiring extensive mathematics. Even for alternatives to the calculus-directed sequence, such as statistics or computer science, high school algebra provides a beneficial background.

Algebra for these students should include algebra of matrices as related to transformations of the plane and should have a strong analytic geometry component. In addition, the advanced algebra topics should include exponential, logarithmic and trigonometric functions.

Measurement

Except for the practical usefulness in application, measurement as a mathematical topic might be included in the relations strand as a function which maps a set of numbers to a property of an object or a set. However, the importance of measurement in the world of the consumer as well as in a multitude of business and scientific applications demands consideration as a separate strand.

Measurement, at the simplest level of application, is such an integral part of daily life that the inability to measure and make estimates of measurements is a serious handicap. Whether one is telling time, cutting and fitting a pattern for a dress or bench, mixing a cake or cement, skills of selecting and using appropriate units of measurement are basic.

At a slightly higher level are many technical careers which depend on more refined skills of measurement and estimation. Such careers include mechanics, carpenters, tailors, drafters and various construction workers. Beyond this technical use are those scientific and technological professions which require sophisticated use and design of high precision tools of measurement as well as measurement systems to meet the demands of an increasingly complex technological, space-oriented society.

Fundamentally, measurement should be taught as a process independent of specific units. The concept of measurement should be developed intuitively from rough comparisons of the property being measured (time, volume, weight, length). Initially, these comparisons (longer, greater, heavier) should be between two objects that have the property being measured. Then a need for standard units should arise out of the use of nonstandard units.

In the development of measurement concepts, the approximate nature of measurement should be emphasized. A property such as length is continuous, and the assignment of a number by counting the number of times a unit is contained in a length is deceptive because of the association of counting with the determination of the exact number of discrete objects in a set-cardinality sense. The difference between precision and accuracy should be emphasized at all levels and in the context of measurement as an approximation.

Estimation in a measurement setting probably has more routine and daily applications for most people than has the use of precise measuring tools. Out of measuring experiences with common units, especially metric units, students striving for at least a minimal level of achievement should develop a feeling for units of length, area, volume, time, weight and capacity to enable them not only to select and use units appropriately but also to estimate measurements with confidence. For these students, the practical uses of measurement cannot be taught as a textbook topic. Students must have experience in measuring, estimating measurements and checking their estimates at a variety of precision levels.

An equally important use of estimation in a nonmeasurement setting involves making an estimate to judge the reasonableness of an answer in the solution of a problem or even in the computation of a result. Certainly in any problem setting, whether it involves measurement or not, emphasis should be placed on always making such a judgment about reasonableness of results.

Those students who expect to continue their education in vocational areas which depend heavily on measurement, should have sufficient understanding to go beyond the routine daily applications of measurement. Extensions of this strand for students striving for attainment of Collection P objectives should include the study of rates of change, scientific notations, significant digits as well as algebraic representations in problem applications.

For those students working toward Collection E objectives, measurement should be treated under the more general topic of function as correspondences which associate number with theoretical point sets and subsets. This treatment should be theoretical, but should not exclude the same kind of practical experiences suggested for attainment of Collections C and P.

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Probability and Statistics

Inclusion of probability and statistics in the secondary curriculum has been proposed by many mathematics educators and mathematicians for the past decade. The subject of the first International Comprehensive School Mathematics Program (CSMP) conference held in 1969 was on the teaching of probability and statistics at the precollege level. The recommendations suggested that all students would benefit from these subjects taught starting from a wealth of realistic examples and using practical experiments as well as simulation methods.

Probability and statistics are two distinct but related fields. Probability is a theoretical concept that measures the likelihood of the occurrence of chance events. Some of the uses of probability are largely informal and intuitive. These involve rough estimates of such events as rainy weather or of passing a test. However, a precise probability estimate is considered as the proportion of times a particular event will occur if the circumstances surrounding the event are repeated indefinitely.

Statistics, on the other hand, is concerned with the evaluation of data collected under real circumstances. Descriptive statistics involves the organization and presentation of data in a manageable and interpretable form. Inferential statistics is concerned with the extent to which knowledge of the characteristics of sample data can be used to estimate the characteristics of an entire population. However, inferences about a population based on the information derived from a sample cannot be made with certainty. Thus, the conclusions of inferential statistics must be stated as probabilities.

There seems to be little question that every person should have some experience with both descriptive and inferential statistics as well as probability. Every citizen who plans to vote, take out insurance, read the daily paper or watch TV needs to have some awareness of basic probabilities, of the limitations of inferences drawn from data and of the ways information may be distorted either intentionally or through ignorance.

For productive citizenship, students should have exposure to ideas beyond the limited descriptive statistics that has previously been included in mathematics curriculum at the junior high level. Students working toward Collection C objectives should at least experience probability and statistics at the minimal level recommended at the CSMP conference.

Topics appropriate for Collection C objectives can be developed in an experimental mode using simulation as well as practical examples with the concepts presented informally and intuitively. All students should have enough experience with the basic ideas of probability and statistics so that they are not duped by commercial claims, they can interpret simple probability statements (such as weather reports) and they can read, interpret and draw valid inferences from data presented in a table or graph.

In addition to the practical, intuitive probability and statistics appropriate for all students, those striving toward Collection P objectives should be introduced to a more formal treatment of such fundamental ideas as measures of dispersion, basic counting principles, permutations and combinations and the notion of correlations with an emphasis on uses and abuses of these and other statistical procedures.

Students who plan postsecondary study that requires extensive mathematics can benefit from a more theoretical presentation of probability and statistics. Such topics as mathematical expectation, conditional probabilities, notions of sampling, inferential statistics and hypothesis testing are appropriate for these students. A balance between theory and application should be maintained. Certainly they should not be deprived of *hands on* and simulation experiences but should also encounter more formal, theoretical treatments.

Computing and Computers

Recent NCTM, NCSM and MAA position papers on basic skill areas in mathematics have included statements regarding computer use and computer literacy. Computer literacy refers to the nontechnical or low technical aspects of the capabilities and limitation of computers as well as to their social, vocational and educational implications. Computer literacy requires a knowledge of how to use a computer, an awareness of its products (and by-products) and a knowledge of applications of computers to a variety of tasks.

In today's world everyone's life is touched by the computer. The reality of this fact combined with the growing employment demand in the computer industry places an increasing responsibility on the education community to provide students with the necessary background to be aware of and participate in careers contingent upon various levels of computer literacy and use.

No other topic in this guide will serve to date it more than an attempt to address comprehensively the topic of computing devices. As time passes, the state of the art as described here will undoubtedly be made humorous (even ridiculous) by continuing developments in the electronic industry. There continue to be daily surprises from the industry — more computer memory available in smaller packages, more programming features available for hand-held calculators, greater availability of personal computing as well as office/business/school computing. All of these offerings are being made available at rapidly decreasing cost.

Historically, few machines have had an impact comparable to that of the electronic computer. Indeed, if its use continues to grow as fast as it has since its inception in the 1940's, the computer may well earn a place in history comparable to, or even surpassing, that of Gutenberg's press.

Clearly, there are no longer distinct definitional differences between calculators and computers. There are hand-held calculators presently on the market that accept modular packages of computing routines — using the calculator only as a data entry device — and plug-in printing machinery so that permanent results are available. The terms *micro* and *mini* seem doomed to fade into irrelevance as minicomputers have already taken on the attributes of *large* computers and microcomputers can now do the tasks formerly possible only with minicomputers and computers.

The state of the teaching art in the field is even less predictable. However, Georgia's responsibility to its secondary school students goes substantially beyond the NACOME recommendation of permission to use hand-held calculators in classrooms. Computer literacy is useful now and will be essential in future years. Admittedly, the state of the art changes so rapidly that instruction in literacy is difficult, but delay now will only mean much more work when such instruction is begun.

The mathematics curriculum is the proper place for a computer literacy program to begin because mathematics and the study of computers complement each other. The problem-solving approach used in mathematics can serve as a model for the development of computer programs and the computer can serve as a tool to motivate and improve the study of mathematics. In this complementary relationship, the computer should not dominate or dictate the curriculum. Rather, the computer should serve as a means of aiding the achievement of existing objectives upon which the mathematics program is built.

The immediate implications for the classroom teacher are that the number of calculations, the size of the numbers used or the tedium of computation should no longer be barriers to the solution of interesting problems. New technology can help teachers develop students who can learn on their own, solve problems, understand algorithmic processes and make significant applications of mathematics. The availability of computing tools will give teachers the opportunity and time to help students learn mathematics that is either taught poorly or not at all in many classrooms today. Furthermore, it can remove the drudgery of long computations that tend to block the solution of practical problems.

Another significant implication of the increasing availability of computing tools will be the need to reassess the cost-effectiveness and practicality of spending a large proportion of the elementary mathematics curriculum time on teaching computation.

Mathematical Reasoning and Logic

Basic logic skills and frequently used logical patterns should be identified and taught to all students. The development and transfer of these basics should not be left to chance and should not be mere by-products of the study of mathematics. Their realization should be one of the focal points in a mathematics curriculum.

Traditionally formal logic is the bedrock on which mathematical proof rests. The formal use of logic in reasoning and argument has always been an essential component of classical education.

In a less formal context everyone should be able to construct simple arguments which are basically informal proofs. The essence of these arguments is the ability to infer a statement from other statements and to give reasons for the inference. Plausible reasoning is necessary for intelligent citizenship. Collecting information, making observations, examining consequences and arriving at a decision are reasoning skills necessary to such activities as serving on a jury, voting, investing, buying and solving a variety of problems.

Mathematics should be the setting in which reasoning of this type is taught to every student. Those students who plan to continue into postsecondary studies need a more in-depth study of logic. Because of the difficulty and the amount of time it takes to teach reasoning, there is a temptation to resort to *cookbook* methods and memorization. It is essential that the study of elementary logic at least include tests for validity of arguments, the use of existential and universal quantifiers and their negations. These relate directly to the notions of sets and are graphically represented in Venn diagrams. In addition, for those students working toward Collection P objectives, there should be some exposure to use of syllogisms in a proof setting as well as the hypothetical form of inference (if . . ., then . . .).

For those students who plan postsecondary study that requires extensive mathematics, the mathematics curriculum should provide formal work in set theory and symbolic logic. These students should have extended experience in the application of logic to proof in mathematics including the use of quantified statements, logical connectives, the role of the counter-example, the function of an existence theorem, to mention a few. Intensive study in elementary logic should be an integral part of the mathematics curriculum for talented mathematics students.

Objectives

The objectives included in this guide are the same as those listed in *Essential Skills for Georgia Schools* (1980). The arrangement of the objectives differ only in that in this guide they are written in three collections to assist in using them to design course plans. In grades nine through 12 the mathematics strands or topics are intertwined; therefore, each objective is keyed to one or more strands. The strand codes are indicated, and the objectives are categorized into three collections as described below.

Objectives for Collection C, P and E

Collection C:	The knowledge, skills and processes presumed necessary for productive citizenship.
Collection P:	The knowledge, skills and processes presumed necessary for postsecondary studies such as vocational education or areas of liberal arts not requiring concentration in the field of mathematics.
Collection E:	The knowledge, skills and processes presumed necessary for postsecondary studies requiring extensive mathematics.

Codes for Strands

S/N/N - Sets, Numbers and Numeration
R/F - Relations and Functions
O/P/N - Operations, Their Properties and Number Theory
G - Geometry
A - Algebra
P/S - Probability and Statistics
M/E - Measurement and Estimation
C/C - Computing and Computers
MR/L - Mathematical Reasoning and Logic

COLLECTION C

The learner will

1. classify elements of a set according to common characteristics. (S/N/N, R/F)
2. identify one-to-one, one-to-many correspondences. (S/N/N, R/F)
3. order any given set of rational numbers (whole numbers, fractions, decimals, percents, negative numbers). (S/N/N, R/F)
4. Identify and use different representations of the same number of quantity (including measurement) and translate from one representation to another, for example $1/4 = 0.25$. (S/N/N, C/C, M/E)
5. determine when and how to use the four arithmetic operations. (O/P/N, C/C, R/F)
6. compute efficiently — both with and without a calculator — using whole numbers, fractions, decimals, percents and negative numbers. (C/C)
7. identify prime numbers; find factors and multiples of given number. (S/N/N, O/P/N, C/C, R/F)
8. apply the distributive property of multiplication over addition without necessarily identifying the term *the distributive property*. (O/P/N, C/C)
9. apply the associative and commutative properties of addition and multiplication without necessarily identifying the terms *associative* and *commutative*. (O/P/N, C/C)
10. select the order of arithmetic operation necessary to simplify a mathematical expression or to solve a real-world problem. (O/P/N, C/C)
11. use estimation, i.e., calculate with rounded numbers if the situation can be satisfied with an approximate answer — using both mental and written calculation. (S/N/N, R/F, O/P/N, M/E, C/C)
12. judge reasonableness of answers. (O/P/N, A, G, P/S, M/E, C/C, MR/L)
13. describe effects, uses and limitations of computers in society. (C/C)
14. find a rule (relation) when some pairs of numbers are given and find pairs of numbers when a rule (relation) is given. (R/F)
15. identify and classify sets of points including points, lines, planes, three dimensional figures, line segments, open curve, closed curves, angles, triangles, rectangles, squares and circles. (G)
16. identify shapes that are alike if stretching, shrinking or bending is allowed and cutting or joining is not allowed. (G)
17. identify shapes that are alike under rotations, reflections or translations. (G)
18. identify relations between points sets or between geometric figures such as parallel, perpendicular, similar and congruent. (S/N/N, R/F, G)
19. read and make scale drawings. (R/F, G, M/E, C/C)
20. locate points in a Cartesian plane. (R/F, G)
21. perform geometric constructions using
    a. straightedge and compass (as measuring instrument)
    b. ruler and protractor. (G, M/E)
22. find missing measures of sides and angles of geometric figures using relationships such as Pythagorean Theorem and properties of similar figures. (G, M/E)
23. apply standard formulas including those for perimeter, area, volume, circumference, time-rate-distance, interest and selling price. (G, M/E, R/F, C/C)
24. select appropriate units of measurement to determine length, area, volume, perimeter, circumference, angle, time, mass, temperature and capacity. (R/F, M/E)
25. select and use the appropriate instruments to measure length, mass, angle, temperature, capacity and time. (M/E)
26. estimate measurements with a reasonable degree of accuracy. (M/E)
27. determine the precision of measurement required for a given situation and be able to select the unit required for the precision. (M/E)
28. collect and organize data. (S/N/N, R/F, P/S)

29. construct and interpret graphical representations such as tables, charts, graphs, maps and histogram. (R/F, P/S, C/C, M/E)
30. read and interpret diagrams including simple flow charts, tree diagrams, factor trees and Venn diagrams. (S/N/N, R/F, P/S, C/C)
31. illustrate how sampling may affect interpretation of data. (P/S, C/C)
32. assign or estimate the probability or odds of a chance event. (S/N/N, P/S)
33. use observations and data to make predictions. (P/S)
34. translate a real world situation into problem(s) that reflect the situation and apply mathematics to that (those) problem(s) where appropriate. (ALL)
35. compute the probability P(~A) when given an event A and P(A). (P/S, C/C)
36. determine the range, mean, median and mode of both collected and give data and recognize uses and misuses of these terms in the interpretation of data. (P/S, C/C)
37. read and interpret materials, e.g., textbooks, forms, references, coded charts, requiring the use of mathematics at the appropriate level. (ALL)
38. read, interpret and complete forms pertinent to personal finance, employment and citizenship responsibilities, and apply mathematics to develop a plan for spending, borrowing and investing. (ALL)
39. use mathematical reasoning to express and support a point of view. (MR/L)
40. make and interpret generalized statements using *all*, *some*, *none*, *or* and *and*. (S/N/N, R/F, MR/L)
41. analyze arguments critically, recognize common errors in reasoning and exhibit critical thinking. (S/N/N, R/F, MR/L)
42. exhibit an awareness of the need to acquire a broad range of mathematical skills in order to enter into careers or fields such as technology, physical sciences, behavioral sciences, biological sciences, economics and commerce. (ALL)
43. exhibit an awareness of the contributions that mathematics has made and is making in the cultural development of civilization. (ALL)

COLLECTION P

The learner will

1. give equivalent forms of rational numbers in
   a. exponential form,
   b. scientific notation,
   c. decimal notation, including repeating nonterminating decimal notation. (S/N/N)
2. recognize the absolute value of a real number as the distance that number is from 0 on a number line and solve simple equations and inequalities involving absolute values. (S/N/N, R/F, M/E)
3. give the major classifications of the complex number system with emphasis on the real number system, and identify the subsystem(s) to which any given number belongs. (S/N/N)
4. illustrate the concepts of betweenness and density with the use of a correspondence of the rational numbers and points on the number line. (S/N/N, R/F)
5. determine, when given a relation defined by a rule, a mapping, a set of ordered pairs, or a graph
   a. the domain,
   b. the range,
   c. whether or not the relation is a function. (R/F)
6. identify and apply the following properties of equality or inequalities
   a. reflexive,
   b. symmetric,
   c. transitive,
   d. substitution,
   e. trichotomy. (R/F)
7. graph the set of points on a number line or the Cartesian coordinate plane which satisfy simple given conditions. (R/F)
8. compute efficiently using rational and irrational numbers. (O/P/N, C/C)
9. perform efficiently the four basic operations on algebraic expressions. (O/P/N, C/C)
10. evaluate algebraic expressions when given a value for a variable. (O/P/N, C/C)
11. identify and apply the associative, commutative, distributive and closure properties, and identify the identity and inverse elements for a given operation defined over a given set. (R/F, O/P/N, C/C)
12. determine if a given number is divisible by 2, 3, 4, 5, 6, 9 or 10 by applying rules of divisibility without actually calculating a quotient. (R/F, O/P/N, C/C)
13. factor any given number into its unique product or prime numbers (Fundamental Theorem of Arithmetic). (R/F, O/P/N, C/C)
14. demonstrate the ability to
   a. derive and apply formulas for the area and perimeter of parallelograms and triangles;
   b. apply formulas for area and perimeter (or circumference) or other two-dimensional figures;
   c. apply formulas for surface area and volume of three-dimensional figures such as cones, spheres, pyramids, cylinders and rectangular solids. (O/P/N, G, M/E)
15. identify and classify the following
   a. triangles (acute, obtuse, right, scalene, isosceles, equilateral),
   b. quadrilaterals (parallelogram, rectangle, rhombus, square, trapezoid),
   c. other polygons (pentagons, hexagons, octagons),
   d. special polygons (equiangular, equilateral, regular),
   e. polyhedra (rectangular solids, cubes, prisms, pyramids, tetrahedron),
   f. other space figures (cones, cylinders, spheres). (S/N/N, R/F, G, M/E)
16. use Venn diagrams to illustrate
   a. union of sets,
   b. intersection of sets,

c. complement of a set,
d. disjoint sets. (S/N/N, R/F, G)

17. identify a network as path between vertices, odd and even vertices and determine if the network is traversable. (R/F, G)
18. compute accurately and efficiently with signed numbers. (A, C/C)
19. translate a simple real world situation into an expression, equation or inequality involving variables. (R/F, O/P/N, A)
20. solve linear equations and inequalities using the addition, subtraction, multiplication and division properties of equality or inequality. (R/F, O/P/N, A)
21. identify a system of equations as dependent, independent or inconsistent and solve the system by the following methods
    a. substitution,
    b. addition and subtraction,
    c. graphing. (R/F, A, C/C)
22. interpret slope in a linear equation as a rate of change. (R/F, A)
23. transform a given linear equation into the slope-intercept form, determine slope and the y-intercept and use them to plot the graph of the equation. (R/F, A, C/C)
24. write an equation on a line when given
    a. its slope and y-intercept,
    b. its slope and any point on the line,
    c. its x- and y-intercepts,
    d. any two points on the line. (R/F, A, M/E)
25. identify the type, degree and coefficients of a given polynomial and be able to add, subtract, multiply and divide given polynomials. (A, C/C)
26. factor polynomials which include one or more of the following types
    a. those having a common monomial factor,
    b. difference of two squares,
    c. sum or difference of two cubes,
    d. perfect square trinomials,
    e. general trinomials. (A, C/C)
27. solve simple quadratic equations by factoring. (A, C/C)
28. write an equivalent form for a given algebraic expression which includes radicals, exponents or algebraic fractions. (A)
29. derive a valid conclusion from a set of logically related sentences (whenever possible) and test the validity of a proposed conclusion to such a set of sentences. (MR/L)
30. state the converse and biconditional related to a given conditional statement and use the terms "necessary, sufficient and equivalent" in describing the relationships between the conditions involved in these statements. (MR/L)
31. state the contrapositive of a given conditional statement. (MR/L)
32. form conjunctions, disjunctions and negations of given sentences. (MR/L)
33. determine if a given distribution approximates a normal distribution. (P/S)
34. illustrate that two sets of data may have the same range, the same spread and the same mean, but not necessarily all three. (MR/L, P/S)
35. apply basic counting ideas such as fundamental counting principles, permutations and combinations, to given problem situations. (A, P/S, C/C)
36. find for events A and B, the P(A and B) and P(A or B) for independent events A and B. (S/N/N, P/S)
37. illustrate events A and B which are dependent. (S/N/N, P/S)
38. illustrate events A and B which are mutually exclusive. (S/N/N, P/S)
39. summarize data by
    a. constructing a frequency distribution;
    b. constructing a graphical representation;
    c. calculating measures of central tendency — mean, median, mode;

d. calculating measures of dispersion — range, variance, standard deviation. (R/F, P/S, C/C)

40. interpret and compute percentile scores in a set of data. (P/S, C/C)

41. illustrate correlation in paired data, e.g., Scatter Diagrams. (R/F, P/S)

42. use observations and data to formulate relative frequencies and compare them to given probabilities that apply to the data and/or use to estimate probabilities. (R/F, P/S)

43. find the mathematical expectation given probabilities and the payoffs for given events. (P/S)

44. follow a flow chart or algorithm and determine the outcome. (MR/L, C/C)

45. illustrate differences among levels of computers — from hand-held calculators to large scale computers. (MR/L, C/C)

46. use chained operations with a hand-held calculator, correctly applying the hierarchy of arithmetic operations. (MR/L, C/C)

47. use a calculator efficiently in applications to other disciplines. (ALL)

48. estimate area and capacity measurements with a reasonable degree of accuracy. (M/E)

49. perform geometric constructions using straightedge and compass. (G, M/E)

50. estimate angle measurements with a reasonable degree of accuracy. (G, M/E)

51. estimate the average of a set of five or fewer numbers. (P/S, M/E)

52. estimate time needed for completion of a task and for travel over a specified course. (M/E)

53. estimate answers to multiplication and division problems using scientific notation. (S/N/N, M/E)

54. estimate square roots. (M/E)

55. determine largest unit of error for a given measurement instrument. (M/E, C/C)

56. determine number of significant digits in a measurement. (M/E)

57. estimate equations of straight lines given a graph of a line(s). (A, M/E)

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COLLECTION E

The learner will

1. identify characteristics of subsets of complex numbers, e.g., real, irrational, rational, integers, counting, in relation to such properties as closure, order, density and completeness. (S/N/N, R/F, O/P/N)
2. construct and analyze proofs in set theory. (S/N/N, MR/L)
3. write and read numerals in different bases including bases two, eight and ten, as well as nonintegers. (S/N/N, O/P/N, C/C)
4. exhibit a knowledge of the definition, notation and pictorial representations of set theoretic concepts including elements, subsets, universal set and null set and exhibit a knowledge of set operations such as union, intersection and Cartesian products. (S/N/N, R/F).
5. state the characteristics of a partition and illustrate by example the variety of applications of this concept, e.g., partitioning regions, classification schemes (counting problems, equivalence classes). (S/N/N, R/F)
6. determine the following, given a rule for an operation and a set on which the operation is defined
   a. an operational table,
   b. identity and inverse elements,
   c. the existence of closure,
   d. the existence of properties, such as commutative, associative and distributive. (O/P/N, R/F)
7. apply definitions to
   a. check for algebraic constructs such as group, field, and finite geometrics — given an operational table (such as a modular system), sets of numbers or sets of points,
   b. construct a group, field, finite geometry or other system. (O/P/N)
8. apply definitions from number theory such as abundant, perfect or triangular to numbers in bases and in other operational systems such as modular systems. (O/P/N)
9. compute limits of sequences. (O/P/N, C/C)
10. write a given finite series in a sigma notation, and write the finite series given a sigma notation. (O/P/N, A)
11. determine convergence or divergence for infinite series using common tests and compute infinite sums where appropriate. (O/P/N, C/C, A)
12. use mathematical induction in proving theorems. (O/P/N, MR/L)
13. use definitions or divisibility, and apply the Fundamental Theorem of Arithmetic in proving theorems. (O/P/N, MR/L)
14. identify and/or define
    a. relation,
    b. domain,
    c. range,
    d. function,
    e. equivalence relations,
    f. composition of function,
    g. inverse function. (S/N/N, R/F)
15. identify and/or define the following functions
    a. identity,
    b. constant,
    c. absolute value,
    d. greatest integer,
    e. piece-wise defined, e.g., $f(x) = g(x)$ if $x < a$
       $u(x)$, $a \le x \le b$,
       $v(x)$, if $x < b$. (O/P/N, R/F)

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16. describe a mapping such that
    a. the mapping is a function without an inverse,
    b. the mapping is a function with an inverse. (R/F)
17. determine the following for a mapping or mappings
    a. the image of a given domain value,
    b. the pre-image of a given range value,
    c. the inverse of the mapping,
    d. the composites of the mappings. (R/F)
18. solve and graph
    a. linear and quadratic equations,
    b. linear and quadratic inequalities,
    c. absolute values (equalities and inequalities),
    d. systems of equations and inequalities. (R/F, A)
19. apply the concepts of relations and functions to the following
    a. linear programming,
    b. inverse and direct variation,
    c. vertical line test for f and horizontal line test for f'. (R/F, A)
20. discuss given functions in terms of
    a. symmetry,
    b. continuity,
    c. asymptotes,
    d. slope,
    e. rates of change,
    f. intercepts,
    g. maximum,
    h. minimum,
    i. boundedness,
    j. ultimate direction,
    k. exclusions from the domain,
    l. intervals in which zeroes occur. (R/F)
21. describe geometric transformations in terms of functions. (R/F, G)
22. identify some relations in a Cartesian product A × A as partitions of A. (A)
23. transform into standard form and sketch the graphs of equations of conics. (A)
24. derive an equation given its roots or derive a function given points of its graph. (A)
25. find the points of intersection of conics with straight lines and conics with conics such as
    a. straight lines and parabolas,
    b. straight lines and circles,
    c. two parabolas,
    d. two circles,
    e. parabolas and circles. (A, G)
26. state and use the following theorems
    a. Rational Zeroes Theorem;
    b. Factor Theorem
    c. Remainder Theorem,
    d. Fundamental Theorem of Algebra. (O/P/N, A)
27. demonstrate the relationship between exponential (base 10 and e) and logarithmic functions through
    a. defining,
    b. translating their equations,
    c. graphing. (O/P/N, A)
28. use laws of logarithms to solve equations. (A)
29. define trigonometric functions in terms of the unit circle and right triangle, and graph these functions. (R/F, A)

30. apply trigonometric definitions, identities and laws to problem solving situations. (A, C)
31. apply the Binomial Theorem to appropriate situations. (A)
32. perform basic operations with vectors and apply these to problem situations, e.g., geometry and physics problems. (A, G)
33. use the algebra of matrices and recognize the convenience this notation provides in modeling and solving problems, e.g., solving systems of equations. (R/F, A)
34. perform basic computations involving complex numbers, solve equations having complex roots and graph complex numbers. (A, C/C)
35. solve and graph equations written in polar form. (R/F, A)
36. determine probabilities of events with emphasis on the following notions
    a. sample space,
    b. random variable,
    c. mathematical expectations,
    d. dependent and independent events,
    e. conditional probability and Bayes' Theorem. (S/N/N, P/S)
37. identify probability distributions including binomial and normal and recognize the distribution as a function whose domain is a set of events, x, and whose range is a set of probabilities, P(x). (S/N/N, R/F, P/S)
38. draw inferences about population parameters from sample statistics using hypothesis testing or estimation procedures. (P/S, MR/L)
39. use techniques to construct a representative or random sample from a given population. (P/S, MR/L)
40. calculate and interpret a measure of correlation for paired data. (R/F, P/S, C/C, M/E)
41. recognize that an axiomatic system is characterized by undefined terms, assumptions, definitions and theorems and that theorems are derived from these assumptions through the application of logical reasoning. (ALL)
42. apply inductive and deductive reasoning to develop and to prove theorems from the following approaches
    a. synthetic geometry,
    b. coordinate geometry,
    c. transformational geometry. (G, MR/L)
43. solve problems and write proofs using definitions, axioms and theorems including theorems dealing with congruency, similarity, Pythagorean relationships, geometric inequalities, parallelism, perpendicularity, angles and arcs. (MP/L, A, G)
44. describe loci and perform standard geometric constructions including congruent figures, parallel lines, perpendicular lines, bisectors and circumscribed and inscribed polygons. (R/F, G)
45. determine if a network can be traversed. (G)
46. determine the order of a given network. (G)
47. determine if a Hamiltonian path exists for a given network. (G, MR/L)
48. read a flow chart utilizing iteration techniques and determine the outcome of the charted procedure. (MR/L, C/C)
49. prepare a flow chart using
    a. iteration techniques,
    b. branching from conditional statements. (MR/L, C/C)
50. apply programming to support another branch of mathematics or natural science. (MR/L, C/C)
51. simulate a common algorithm, e.g., long division or fraction reduction with a work description, flow chart and program. (MR/L, A, C/C)
52. construct indirect proof. (MR/L, A, G)
53. determine equivalence between sentences involving conjunctions, disjunctions, negations and conditionals (do not restrict such discussions to truth tables). (MR/L)
54. classify statements as being claims of existence, uniqueness or universality and determine the preceding objective above. (MR/L)

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55. determine the truth tables for sentences involving the connectives of conjunctions, disjunctions, negations, conditional, biconditional and combinations thereof. (MR/L)
56. use Venn diagrams to illustrate the relationships represented by the truth tables of the preceding objective above. (S/N/N, MR/L)
57. represent sentences in symbolic form and use the tools of truth tables and symbolic logic in assessing equivalence and validity. (MR/L)
58. determine function errors (dy) corresponding to deviations in domain value (dx). (R/F, M/E)
59. use inequalities and absolute values to describe lengths, areas and volumes. (R/F, M/E)
60. apply the distance function to describe planar areas such as circular discs and volumes such as spheres. (R/F, M/E)

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Sample Course Descriptions

A selection of sample course descriptions is included in this section for committees or individual teachers to use while developing courses of study for their secondary school or school system mathematics program. This set of sample course descriptions is *not* a complete set of courses to be offered in a school. Some course samples *may not* be appropriate for a given school. Courses for a school or school system must be identified and planned on the basis of the needs of the given student population.

The course samples vary in length — quarter, semester and year. The length indicated for a given course is not intended to be a recommendation. A course or a set of courses can be redesigned at the local level to accommodate the quarter, semester or yearly organization of the school system.

Types of activities are suggested for each course. The activities should be specified at the local school or system level. The activities selected will depend upon the interests, abilities and needs of the students and the resources of the school. A few sample activities are included within each course description. Nonroutine problems are provided to serve as a starter for a growing collection of problem-solving activities to permeate the entire mathematics curriculum. The reference list contains numerous sources for building a collection of problems in the various topics of study.

Sample Course Plan

Course title — *General Mathematics (year)*

(This course may be organized by designing minicourses and arranging them according to need of students.)

Course description

This course includes estimation and measurement skills; computation skills with decimals, percent and common fractions; informal geometry; using formulas; collecting data, constructing tables to display data and reading and interpreting tables and charts; applications of the above skills in problem situations.

Course objectives

The student will

1. make reasonable estimates of length, area and volume.
2. make reasonable estimates of sums and products of two whole numbers up to three digits in length.
3. identify shapes that are alike if stretching, shrinking or bending is allowed.
4. make accurate measurements by using measurement devices.
5. compute efficiently — both with and without a calculator — using whole numbers, fractions, decimals, percents and negative numbers.
6. select the order of arithmetic operations appropriate to a given situation.
7. identify basic geometric shapes, including parts such as angle, vertex, diagonal.
8. substitute correctly and compute the results of simple formulas, such as area formulas in geometry.
9. solve ratio and proportion equations.
10. apply "ratio and proportion" techniques in practical settings.
11. apply rudimentary concepts of coordinates in practical settings, such as map readings.
12. decide which of two fractions is the smaller.
13. read and make scale drawings.
14. find factors and multiples of given numbers.
15. use the associative and commutative properties of addition and multiplication without necessarily identifying the terminology.
16. order a set of decimals.
17. order a set of negative numbers.
18. use place value to write and name numbers, including decimal fractions.
19. use the value of pi in common area formulas.
20. estimate measurements such as temperatures in various conditions using both Celsius and Fahrenheit scales.
21. collect and organize data.
22. read and interpret tables and charts.

Course content

Basic computation - with and without a calculator

Measurement and estimation of lengths

Estimation and calculation of areas

Measurement and estimation of time and temperatures

Measurement, estimation and calculation of volumes

Informal geometry

Ratio and proportion

Scale drawings

Map reading

Fractions, decimals and percents

Applications of basic arithmetic to problem situations

Common formulas — techniques of substitutions

Instructional activities

(The following activities and many others may be developed at the local level.)

Practicing computation, checking answers with calculators

Using unconventional measurement devices such as micrometer, vernier caliper, decibel meter, volt meter, stop watch as well as the yardstick, meterstick and tapes (See Sample Activity *Using an Ohmmeter — General Mathematics*)

Investigation of scale models or cars, boats or airplanes (In airplane models, are horsepower requirements reduced in the same scale as length measurements?)

Using musical timing to illustrate fractional equivalents

Using road maps, topographical maps, aeronautical charts, weather maps, waterway charts

Using protractors and graph paper to measure angles and areas in informal geometry

Making applications in other fields or in the real world in which students work in teams (See Sample Activity *Physical Science Application — General Mathematics.*) Written or oral reports may be made in some content areas.

- Measurement systems — metric and customary
- Types of measuring instruments
- Specialized instruments — music, medicine, electronics, aviation, boating, automotives

Going on field trips or having visiting speakers

Evaluation

Teacher-made or standardized tests may be used well in the following content areas.

Estimation skills
Calculation skills
Use of scales in drawings and maps
Reading and interpreting charts and tables
Application of arithmetic in everyday problem situations
Identification of geometric shapes and related problems
Interpretation of displays on measuring instruments
Evaluation of written or oral reports
Evaluation of group projects

General Mathematics Sample Activity 1

Activity title — *Using an Ohmmeter*

Activity objective

The student will make accurate measurements by using appropriate measurement devices.

Time

Two to three days will be required for these activities, and preparation of and procurement of materials will require three to five days, depending on the availability of materials.

Materials

3 x 5 cards (1 per packet)

Ohmmeter (1 per packet)

Registers (at least 4 per packet, with resistance varying from 100 to 500,000 ohms)

Register decoding charts (1 per student)

Instruction for ohmmeter (1 per student)

Worksheets with questions and spaces for recording answers

Method

Distribute the materials in individual packets or packets for groups as budgets dictate.

Students should make measurements in the sequence specified by the worksheet supplied by the teacher. (Packets should include instructions for operating the ohmmeter and for deciphering the code on the register.)

Students should verify their measurements by interpreting the codes on the transistors.

All instructions should be very detailed according to the needs of the students — some may be unfamiliar with the equipment.

Evaluation

Questions on register codes or percentages of error tolerance may be included on tests.

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General Mathematics
Sample Activity 2

Activity Title — *Physical Science Application*

Activity objective

The student will

1. make accurate measurements by using appropriate measurement devices.
2. compute efficiently — both with and without a calculator using whole numbers, fractions, decimals, percents and negative numbers.
3. apply ratio and proportion techniques in practical settings.

Time

Three to four days will be required for these activities and related study. Preparation and procurement of materials will require two to three days (and the generosity of some teachers in the science department regarding the use of their equipment).

Materials

Enough arm balances or pan balances to allow students to take turns weighing

1 box of table salt

1 50-ml beaker for each group of students

1 graduated cylinder (50ml) for each group of students

Access to water

Worksheets

Periodic atomic chart (optional)

Method

Prepare a worksheet and an answer sheet for each student. You may wish to ask a science teacher to assist you. The worksheet questions should include the following.

Pour out about a tablespoon full of salt and weigh it on the balance pan. Record the results on the attached worksheet.

The atomic weight of Sodium (Na) is 23, and the atomic weight of Chlorine (Cl) is 35. Salt is sodium chloride NaCl. A molecule of NaCl has weight 58 (23 + 35). What percent of this weight is sodium?

What percent of NaCl is chlorine?

In your amount of salt, how much is sodium and how much chlorine?

If you double your amount, then how much would be sodium?

In 100 grams of NaCl, how much is sodium?

Measure 30 ml of water in the graduated cylinder.

Weigh the 30 ml of water in the beaker on the balance scale. How much does the water weigh?

Using the same technique, weigh 15 ml of water in the beaker on the balance scale.

Compute the weight of 478 ml of water.

Evaluation

Questions concerning perçentage composition, ratio and proportion and metric weights and volumes of water may be included on subsequent tests. At the time of the activity, peer evaluation is appropriate. Also, a science teacher might be invited to submit test items. Performance evaluation of students' use of arm balance is appropriate.

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General Mathematics Problems for Students

A Tennis Tournament

There are N players in an elimination-type singles tennis tournament. How many matches must be played (or defaulted) to determine the winner?

Solution

Each match has one loser, each loser loses only once, so there are N - 1 losers, hence N - 1 matches.

The Handshakers

Every person on earth has shaken a certain number of hands. Prove that the number of persons who have shaken an odd number of hands is even.

Solution

Before any handshakes have occurred, the number of persons who have shaken hands an odd number of times is zero. The first handshake will produce two "odd persons". From then on handshakes will occur between either two even persons, two odd persons, or one odd and one even person. Each even-even shake increases the number of odd persons by two. Each odd-odd shake decreases the number of odd persons by two. Each odd-even shake changes an odd person to even and an even person to odd, leaving the number of odd persons unchanged. Therefore, there is no way that the even number of odd persons can shift its parity; it must always be an even number.

Sample Course Plan

Course title — *Personal Finance (year)*
(This symbol * denotes objectives to be included for a required quarter course.)

Course description

This course includes the basic concepts and skills necessary for being a wise consumer; personal financial planning; aspects of earning and spending money, consumer rights and responsibilities; discussion of credit, insurance, contracts and taxes.

Course objectives — The student will

1. clarify values indicating awareness of how evolving values, goals and resources influence financial decision making.
*2. balance a checking account, given a sample bank statement and returned checks or check stubs, with and without the aid of a calculator.
3. calculate hourly, daily, weekly, monthly and annual wages given the rate of pay for any one of these and interpret standard payroll deduction terms.
4. demonstrate a system of managing personal records of each and credit transactions.
5. interpret rights and responsibilities outlined in credit contracts, warranties and guarantees, sales contracts, rental agreements and lease.
6. identify appropriate resources and consumer procedures in the event of personal financial difficulty.
7. explain the importance of preparing a will.
8. discuss legal aspects involved in writing a will and list the items usually included in a will.
9. estimate costs related to wills and estates.
10. identify financial services available to consumers from four types of financial institutions.
11. compute interest, simple and compound, for a savings account with and without the aid of a calculator.
12. identify and compare various bank services, such as travelers and cashiers checks, overdraft coverage.
13. describe methods of opening a checking and savings account, making deposits and withdrawals from the account.
*14. prepare a realistic budget for a given income, both annually and monthly.
*15. complete an application form for a major credit card and explain the advantages and disadvantages of credit card purchasing.
16. solve problems related to installment plans, credit cards and loans by computing interest, finance charges, down payments, installment payments and total cost.
17. explain how individual credit ratings are determined and identify legal rights of consumer for rating information.
18. compare sources, types, service and costs of consumer credit from four different lending institutions.
19. discuss different types of loans and compute the interest and service charges for any given one.

*20. identify which size of an item is more economical by comparing weights and prices of each.

*21. identify instances of deceptive advertising, packaging, sales persuasion techniques and credit practices.

22. identify various means used to sell goods and services, i.e., advertising, promotions, buying incentives.

23. identify advantages and disadvantages of property ownership; lease agreements and rental contracts.

24. describe or explain a given contract such as rental or sales agreement.

25. calculate costs of maintaining a house or apartment, including monthly costs, such as utility bills as well as maintenance costs such as painting and plumbing.

26. list different types of taxes and state where the funds are used.

*27. solve problems related to taxes by computing income, property and F.I.C.A. taxes.

*28. complete federal and state income tax forms based on given information.

29. define terms associated with insurance (liabilities, exemptions, premium, deductions, net income, disability).

*30. determine the benefits from various types of insurance and describe differences among types of insurance such as term and whole life insurance, property and hospitalization.

31. compute annual and quarterly insurance premiums and compare costs of group and individual plans.

*32. define vocabulary related to investments, such as compound interest, Dow Jones average, common and preferred stocks and bonds.

*33. list types of investments other than stocks and bonds and their advantages and disadvantages.

34. explain methods of purchasing stocks, bonds and mutual funds.

35. discuss the stock broker, his job and fees.

36. compute interest earned for given stocks or bonds and calculate the brokerage fees.

*37. identify sources of retirement income and prepare a sample program for retirement income.

38. discuss tax shelters, pension funds, social security and real estate investments.

*39. identify some agencies dealing with specific consumer problems and cite possible consumer's legal recourse for a given situation.

40. interpret consumer rights and responsibilities involving transactions between individuals and financial institutions.

Course content

Personal considerations
- Values, goals, lifestyles
- Career plans

Earning money
- Employment possibilities and probabilities
- Applying for a job

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Budgeting income
- Allocating available income (long range and short range)
- Estimating monthly expenses
- Deciding on saving and investments
- Adjusting a budget for emergency situations

Consumer information and counseling services and agencies, including independent and government — federal, state and local

Consumer rights and responsibilities

Personal banking
- Checking accounts
- Savings accounts (computation of interest)
- Loans (agencies, payments, computing charges)
- Credit cards (pros and cons of credit purchasing)

Purchasing goods
- Credit rating
- Credit cards
- Cash versus credit
- Costs of credit
- Charge accounts
- Installment buying

Comparison shopping
- Unit prices
- Quality and grades
- Guarantees and warranties

Understanding contracts
- Rental agreements
- Sales agreements
- Insurance policies
- Wills, including writing wills

Renting an apartment or house
- Security deposits
- Signing a lease
- Landlord rights and responsibilities
- Tenant rights and responsibilities
- Cancellation and eviction

Purchasing a house
- Alternatives, the pros and cons of owning condominiums and mobile homes
- Buying to rent or lease
- Obtaining a mortgage
- Down payments, monthly payments, taxes, insurance, closing costs
- Maintenance

Taxes
- Computation of various types of taxes
- Filing various types of forms
- Services provided by tax monies

Insurance
- Personal, e.g., life, hospitalization
- Property, e.g., homeowners, renters, auto
- Job-related, e.g., disability, unemployment, workmen's compensation

Investments

- Stocks (common and preferred), bonds and mutual funds
- Real estate
- Savings — such as accounts and certificate

Retirement

- Retirement income and housing
- Medical considerations

Instructional activities

(The following activities and many others may be developed at the local level.)

Researching various agencies and services available to consumer

Reconciling sample bank statements for a specific time period in which there are checks and deposits, some which remain outstanding

Computing interest earned on various types of saving, e.g., accounts, certificates of deposit including simple and compound interest

Preparing monthly budgets for various given incomes — including monthly expenses, savings and recreation expenses

Comparing newspaper food advertisements. Using a calculator, figure the unit prices for different brand items. Assuming quality is the same, show which products are more economical

Inviting guest speakers from an insurance agency, tax department, bank or savings and loan association, brokerage firm, law office or other pertinent organizations

Completing sample applications for loans and credit cards

Viewing films concerned with banking, consumer information, insurance, taxes, contracts, loans, investments or other applicable topics

From a newspaper ad, listing year, model and price of a car; computing a discount, sales commission, sales tax, down payment, monthly payments for a loan and total cost

For the car chosen, researching insurance costs and determine the lowest possible rate assuming yourself as driver

Comparing term and whole life insurance and discuss other types of life and health insurance

Simulating a case study for a damage suit against a homeowner with details of insurance information given

Categorizing various taxes according to branch of government and state the purpose and use of these tax monies

Completing federal and state income tax forms, (See Sample Activity 1 *Filling Out Income Tax Forms — Personal Finance*)

Field trip to a broker's office to watch the electronic board or ticker tape as transactions occurring on the stock exchanges are reported.

Choosing a well-known corporation and simulate the purchase of a given amount of stock in this particular company. Record daily the closing market price over a six week period and determine amount earned or lost. (See Sample Activity 2 *Investment Comparisons — Personal Finance*)

Planning a sample program for retirement income, including sources of income and housing arrangements

Inviting a lawyer to discuss legal aspects of writing and executing a will

Written or oral reports on major topics

- Insurance — types, differences, costs
- Investments — types, pros and cons
- Consumer agencies and services
- Loans

Evaluation

Teacher-made or standardized tests in several content areas

Computation skills, e.g., interest, taxes
Interpretation of insurance policies, contracts
Purpose and use of various taxes
Application of consumer rights and responsibilities to everyday situations
Identification of differences between certain types of insurances, taxes, loans, investments

Evaluation of written reports or oral presentations

Charts or graphs illustrating results of simulated stock market ventures

Evaluation of assignments, e.g., monthly budget prepared for a given income or preparation of a sample contract, agreement or will

Personal Finance
Sample Activity 1

Activity title — *Filling out income tax forms*

Activity objective

The student will solve problems related to taxes by computing income taxes.

Time

One week may be scheduled for student activities. Teacher preplanning should begin at least a month in advance in order to secure materials.

Materials

Personal income tax forms (1040)

Instruction booklets available from IRS: "Understanding Your Taxes"

Hypothetical W-2 forms for student *wage earner*

Method

Have students fill out tax forms for the following situations.

Short form (no deductions other than personal exemptions)

Long form (with schedules of deductions)

Single

Married, filing separately

Married, filing jointly

Evaluation

On subsequent tests, forms may be filled out for a hypothetical situation.

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Personal Finance
Sample Activity 2

Activity title — *Investment Comparisons*

Activity objectives

The student will

- define vocabulary related to investments, such as compound interest, Dow Jones Average, common and preferred stocks and bonds.
- list types of investments other than stocks and bonds.
- explain methods of purchasing stocks, bonds and mutual funds.
- discuss the stock broker, his job and fees.
- compute interest earned from given stocks or bonds and calculate the brokerage fees.

Time

Approximately 20 minutes per week may be devoted to this activity

Materials

- Poster paper
- Newspapers
- Hand-held calculators

Method

Assign each student a hypothetical sum of $20,000 with the instructions to invest all or part in the stock market, mutual funds, savings account, real estate, gold, futures or other investment. Make certain that careful records are kept weekly charting the growth (or decay) of investments, subtracting appropriate overhead costs, such as brokers fees, real estate fees and interest penalties.

Evaluation

Careful reports should be assessed by the teacher periodically and at the end of the quarter. Profit or loss considerations need not be part of the evaluation; success at investment is not critical in this exercise.

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Sample Course Plan

Course title — *History of Mathematics (quarter)*

Course description

This course is designed to provide a descriptive history of mathematical development from the time of early civilization until the present time. It includes personality sketches of mathematicians as well as history of the development of mathematical concepts. Mathematical techniques of several types are described, but skill in these techniques is not an objective of the course.

Course objectives

The student will—

- write a brief explanation of the concept of numbers.
- describe contributions of early civilizations to mathematical development.
  - Egyptian
  - Babylonian
  - Chinese
  - Hindu
  - Greek
  - Arabs
- identify members of the Greek civilization who contributed to the development of mathematics.
- cite some common applications of mathematics in other fields such as the following.

art	economics
music	government
social science	military
biology	business
physics	school administration
electronics	entertainment

- identify mechanical aids to computation used in the past and at the present time.
- identify some prominent 19th century mathematicians.
- identify some prominent 20th century mathematicians.
- identify branches of mathematics.

Course Content

- The concept of numbers
- Egyptians
- Babylonians
- Chinese and Hindu
- The Greeks — Thales, Pythagoras, Plato, Euclid
- Arabs
- European mathematical development
- American mathematical development

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Application of mathematics

Computing machinery

Mathematics and fine arts

Brief history of geometry

Brief history of algebra

Brief history of higher mathematics

Instructional Activities

(The following types of activities and many others may be developed at the local level.)

Using multitext libraries

Viewing filmstrips, other visual media

Inviting guest speakers

Having trivia quizzes or bees
(See Sample Activity *Trivia Quiz — History of Mathematics*)

Using demonstrations of topological models such as Moebius strips

Using exercises with Greek paradoxes

Having field trips to computer installations to observe the present state of the arts

Group or team projects to develop an oral or media presentation

Role playing lives of prominent mathematicians

Imaginary journalism or "news" reports which hypothetically place students back in history to the times of early mathematical development. (See Sample Activity *Historical Journalism — History of Mathematics*)

Posting photographs with identification of major cultural or mathematical contributions of famous mathematicians at appropriate places in the school

Traditional teacher lecture/discussions

Evaluation

Scheduled teacher-made tests are appropriate for most content areas in this course

Evaluation of student written reports or oral presentations

Evaluation of participation in group projects is also appropriate

College Bowls or tournaments offer a pleasant variety of evaluation employing either team or individual format

History of Mathematics
Sample Activity 1

Activity title — *Trivia Quiz (similar to College Bowl)*

Activity objective

The student will present, state or identify facts related to history of mathematics.

Time

One or two class periods may be used for the actual quiz. One or two weeks of preplanning should be allowed for students designated as resource persons.

Materials

- Print and nonprint materials for research groups
- Index cards (or equivalent material) for recording questions
- Score keeping device

Methods

- Assign a research group of two to four students to construct questions about topics included in (or alluded to) in-class discussion. The teacher should supervise this group checking the questions for depth, relevance, and wording and by checking the answers for accuracy.
- Prearrange teams of four students to participate in the quiz.
- Prearrange a suitable setting and designate a time keeper/score keeper.
- Prearrange for school publicity.
- Schedule rounds for efficient and orderly use of time.

Evaluation

Include several of the most provocative questions on subsequent tests and compare those results with results of other questions.

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History of Mathematics
Sample Activity 2

Activity title — *Historical Journalism*

Activity objective

The student will present in writing facts related to people and events from history of mathematics.

Time

One or two weeks should be allowed for research and writing. One week may be used for classroom sharing. Several weeks after the assignments have been completed may be used for publication of the articles.

Materials

Print and nonprint materials for research

Typewriter

Copying machine (if available)

Method

Make assignments of "courage" of personalities and events from the past that are related to many areas of mathematics.

Allow students to present their reports in class.

Arrange for a method of selecting the best for publication.

Publish the best journal articles by using the school paper or a bulletin board reserved for special items. Local newspapers might allow a series of informative articles that are well-written about mathematical personalities or events.

Evaluation

Students could set up evaluation criteria for evaluation.

History of Mathematics Problems for Students

Problems from Mahāvira

The nature of many of the Hindu arithmetical problems may be judged from the following, adapted from Mahāvira (ca. 850). Solve this problem.

A powerful, unvanquished, excellent black snake with 80 angulas in length enters into a hole at the rate of $7\frac{1}{2}$ angulas in 5/14 of a day, and in the course of 1/4 of a day its tail grows 11/4 of an angula. O ye ornament of arithmeticians, tell me by what time this serpent enters fully into the hole. (Answer: 8 days)

Sample Course Plan

Course title — *Geometry through Constructions and Applications (year)*

Course description

The course is based more on intuitive notions and real world applications than on a formalistic and axiomatic approach. Proof is included but is secondary to other more intuitive considerations. The content is essentially traditional in nature, but the approach to the content is based as much on inductive reasoning as on deductive reasoning. Students investigate theorems through measurement and construction activities. The course begins with an introduction to constructions and inductive reasoning. Problems which involve real-world situations are used to introduce and motivate the learning of theorems whenever possible. In general, the course is designed to promote the perspective that geometry provides a means of describing our world and understanding how to solve problems that occur in our world.

Course objectives — The student will

1. identify and classify basic geometry figures.
2. use compass and straightedge to do basic geometric constructions.
3. use inductive reasoning to reach conclusions.
4. use deductive reasoning to reach conclusions.
5. use congruence assumptions to prove triangles congruent.
6. use basic theorems about triangles, e.g., the Pythagorean Theorem and the Isosceles Triangle Theorem, to solve problems.
7. use basic theorems about parallel and perpendicular lines to solve problems.
8. state properties of trapezoids, parallelograms, rectangles, rhombi and squares and use those properties to solve problems.
9. find the perimeter, area and volume of basic two and three dimensional geometric figures.
10. state the ratio of the sides of a 45°,-45°,-90° triangle and of a 30°,-60°,-90° triangle and use the ratios to solve problems.
11. use basic theorems about inscribed angles, angles formed by tangents, secants and chords to solve problems.
12. use basic theorems about chords, secants and tangents to solve problems.
13. use reflections and rotations to describe properties of geometric figures.
14. find the image of a geometric figure given a rotation, a line reflection, a translation.
15. identify figures with line symmetry with rotational symmetry.
16. use basic similarity theorems to prove triangles similar and to solve problems.
17. use the midpoint, slope and distance formulas to solve problems.
18. apply the relationships of slopes of parallel and perpendicular lines to solve problems.

Course content

Points
- coplanar
- noncoplanar

Lines
- parallel
- intersecting
- concurrent
- skew

Planes
- parallel
- intersecting

Space
- sphere
- cone
- cylinder
- polyhedra

Angles
- acute
- right
- obtuse
- supplementary
- complementary
- vertical

Triangles
- equilateral
- isosceles
- scalene
- acute
- obtuse
- right
- 30-60-90
- 45-45-90

Quadrilaterals
- trapezoid
- kite
- parallelogram
- rectangle
- rhombus
- square

Constructions
- copy circles
- copy segments
- copy angles
- copy triangles
- bisect angles
- perpendicular bisectors
- perpendicular from point to a line
- parallel lines

Circles
- center
- radius
- diameter
- arc
- chord
- tangent
- secant
- central angle

Polygons
- convex
- similar
- congruent

Transformations
- reflections
- rotations
- translations

Perimeter
- triangle
- quadrilaterals
- regular polygons

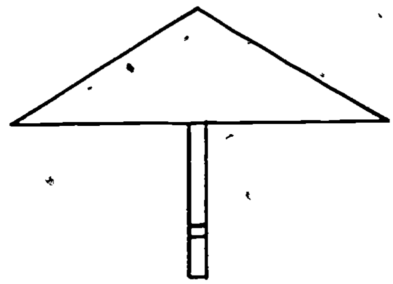
Area
- triangle
- quadrilaterals
- regular polygons
- sphere
- cube

Volume
- cube
- tetrahedron
- right prisms
- cylinders
- cones
- sphere

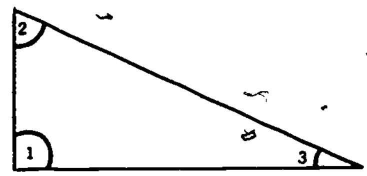
Coordinates
- midpoints
- slopes
- distance formula

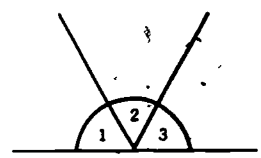
Types of activities

- Activities using concrete materials
  1. An activity can be designed to demonstrate that the intersection of the medians of a triangle is the centroid or balance point of the triangle. Each student is given a sheet of cardstock or cardboard with a triangle drawn on it. The three medians of the triangle are constructed. Punch a small hole in the cardstock at the point of intersection (centroid) of the medians. Cut out the triangular region and place the centroid of the triangle on the point of a sharp pencil. Observe that the triangle balances, that is, it remains parallel to the floor.

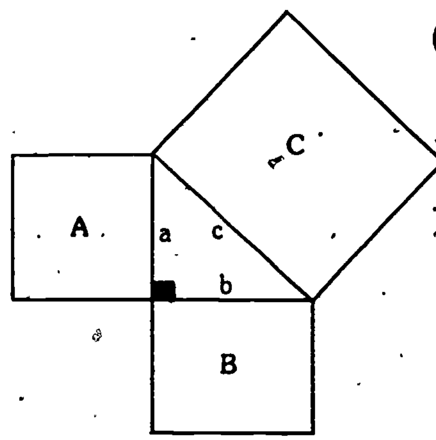


  2. This activity is designed to show that the sum of the measures of the angles of a triangle is 180°. Cut out a triangle from a sheet of paper. Place the "corners" of the triangle to form a straight angle with two sides as shown. The students can observe that $m < 1 + m < 2 + m < 3 = 180^\circ$





  3. There are many proofs for the Pythagorean Theorem. Many involve the concept of area. One approach is to disect squares A and B in such a way to form square C, thus showing a• + b• = c• (See Sample Activity *The Pythagorean Theorem — Geometry through Constructions and Applications)*



- Activities which emphasize inductive reasoning
  1. This activity involves making a table of values and trying to recognize a pattern between the number of sides of a polygon and number of diagonals. The lesson could start by posing the problem, "How many diagonals does a 20-sided polygon have?" Students can determine the number of diagonals for polygons with 3, 4, 5, 6, 7 and 8 sides. Once a pattern is observed the number of diagonals of a 20-sided polygon can be determined.

sides	diagonals
3	0
4	2
5	5
6	9
7	?
8	?

2. A similar approach using a table of values can be used to discover the relationship between the number of sides of a polygon and the sum of the measures of the interior angles of the polygon. (See Sample Activity *Measures of Angles of a Polygon — Geometry through Constructions and Applications*)

sides	measures of angles
3	180°
4	360°
5	540°

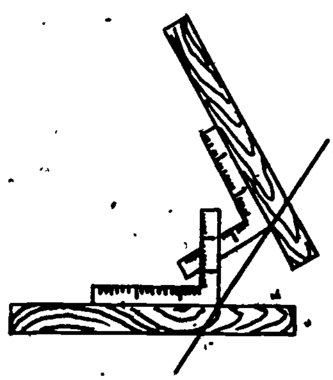
- Activities involving constructions

1. Students enjoy using compasses and straightedges. This activity uses a compass and straightedge to show that the perpendicular bisector of a chord contains the center of the circle. The activity can begin by posing the problem of finding the center of a circle or the center of the arc of a circle. The problem could serve as a motivation for the theorem about the perpendicular bisector of a chord. Students can draw circles and chords and construct the perpendicular bisectors of the chords. They should observe that the constructed lines contain the center.

2. Another construction that students enjoy is finding the circumcenter of a triangle and circumscribing a circle about the triangle. Similarly students can construct the angle bisectors of a triangle and inscribe a circle in a triangle. (See Sample Activity *The Circumcenter of a Triangle — Geometry through Constructions and Applications*)

- Activities involving real-world applications

1. This activity involves a problem that sometimes occurs in carpentry. Suppose two converging timbers are to be cut along a line 1 that a board can be nailed flush against the conveying boards. At what angle should the boards be cut? This problem can be used to introduce the Isosceles Triangle Theorem. Two carpenter's squares can be placed so that BE = DE. The Isosceles Triangle Theorem says that m∡EDB = m∡EBD and hence m∡BDC = m∡DBA. This allows the carpenter to solve a real problem using a basic high school geometry theorem.



2. This activity involves a navigation problem and uses a basic theorem about inscribed angles in a circle. In short, the problem consists of staying outside the arc of a circle. This can be done by keeping the m∡ASB < m∡APB. Note that m∡APB remains constant so long as P is on the arc of the circle. This fact involves the theorem that says, "The measure of an inscribed angle of a circle is one half the measure of the in-

tercepted arc." (See Sample Activity — *Inscribed Angles — Geometry through Construction and Applications)*

Evaluation

Evaluation can consist of paper and pencil test items and whatever other means a teacher elects to use with emphasis on applying theorems rather than proving them. Numerical exercises should be used whenever appropriate as should real-world applications. This approach is consistent with the intent of the course; viz., to rely on an intuitive and inductive approach, rather than a predominantly proof oriented approach.

Evaluation of constructions and drawings

Evaluation of individual and group projects

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Geometry through Constructions and Applications Sample Activity

Activity title — *The Pythagorean Theorem*

Time

One or two class periods may be scheduled for this activity.

Activity objective

The student will discover the Pythagorean Theorem.

Materials

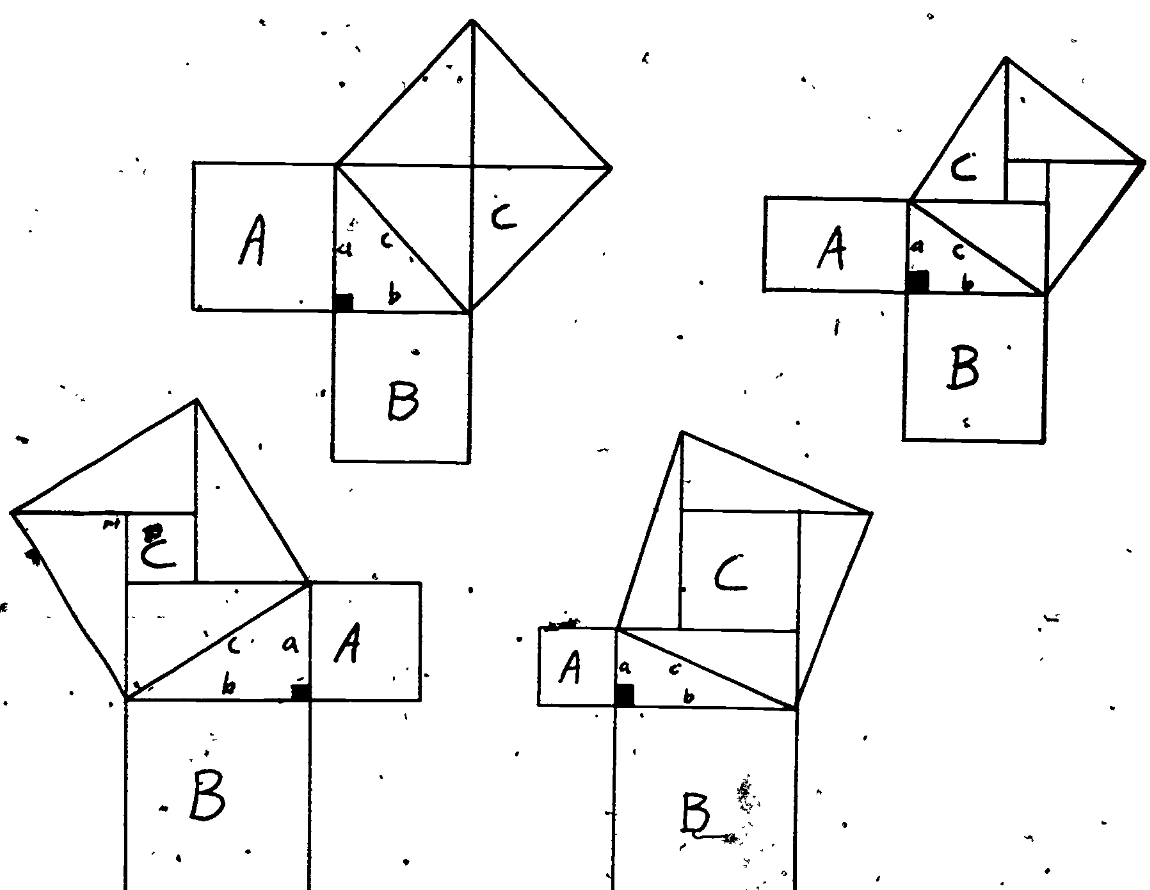
Graph paper with figures drawn as shown below

Scissors

Method

Distribute copies of each of the figures below to students.

Instruct students to cut up the squares A and B and rearrange the pieces to form square C.



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Geometry through Constructions and Applications Sample Activity

Activity title — *Measures of Angles of a Polygon*

Time

One or two class periods may be scheduled.

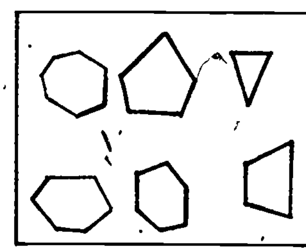
Activity objective

The student will determine that the sum of the measures of the angles of an n-gon is $(n-2) \times 180^\circ$.

Materials

Worksheets with polygons drawn
Protractors

Method



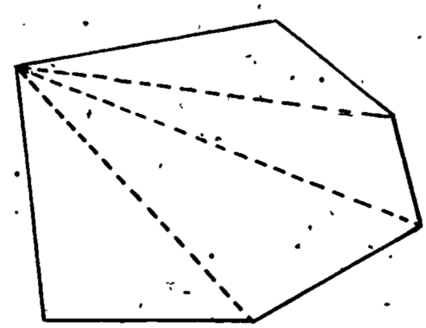
Hand out worksheets with polygons drawn as shown. (The polygons shown are not regular. One could consider using regular polygons.)

Have students determine the sum of the measures of angles of a triangle, quadrilateral, pentagon and hexagon. The sum of the measures could be determined by partitioning the polygons into triangles or by using protractors.

Have students make table to organize data.

Encourage students to predict sum of measures of angles for seven-sided polygon. Determine the sum and see if prediction was correct. Repeat for eight-sided polygon.

Use the data to develop a formula for the sum of the measures of the angles of an n-gon.



4 triangles. Hence 4×180 in interior angles.

sides of polygon	sum of interior angles
3	180°
4	360°
5	540°
6	720°
7	?
8	?

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Evaluation

Draw right triangles and ask students to state the relationship among the sides.

Give the measures of two sides of a right triangle and ask students to find the measure of the third side.

Have students use the Pythagorean Theorem to solve real-life applications.

Given a right triangle, provide an area intersection of the Pythagorean Theorem.

Geometry through Constructions and Applications Sample Activity

Activity title — *Inscribed Angles*

Time

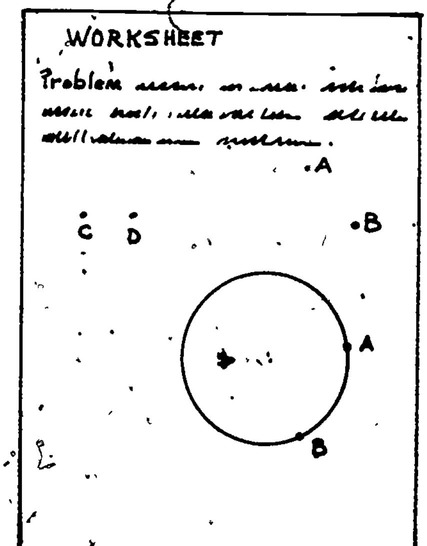
Two class periods may be scheduled for this activity.

Activity objectives

The student will

discover that the measure of an inscribed angle of a circle is one half the measure of its intercepted arc.

apply theorem to a real-world problem.



Materials

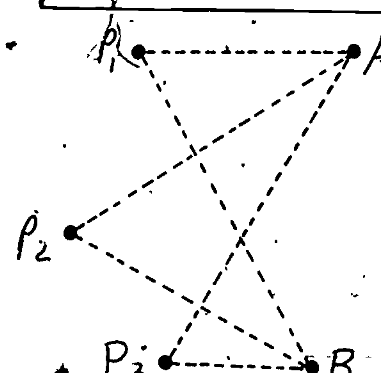
Worksheet for each student

Protractor for each student

Method

Distribute the work sheets which contain the statement of the problem, two sets of two points and a circle with two points marked on it. Discuss and illustrate the problem below.

Problem statement: Suppose the captain of a ship wants to maintain a safe distance from a shoreline. How can the two lighthouses be used to keep a safe distance from the shore? Indicate to students that they will learn a theorem which will help solve the problem.



Using the worksheet have students find points P such that $M < APB = 60°$.

Identify the geometric figures (arc of a circle — actually two arcs of a circle) which the points describe.

Repeat b and c, for points C and D.

Evaluation

Given a polygon, determine the sum of the measures of the angles of the polygon.

Given the sum of the measures of the angles of a polygon, determine the number of sides of the polygon.

Given an n-gon with the sum of the measures of the angles of n-1 angles, find the measures of the remaining angle.

Use the formula to determine the measure of an angle of a regular n-gon.

Geometry through Constructions and Applications Sample Activity

Activity title — *The Circumcenter of a Triangle*

Time

One or two class periods may be scheduled for this activity.

Activity objectives

The student will

determine that the perpendicular bisectors of the sides of a triangle are concurrent.

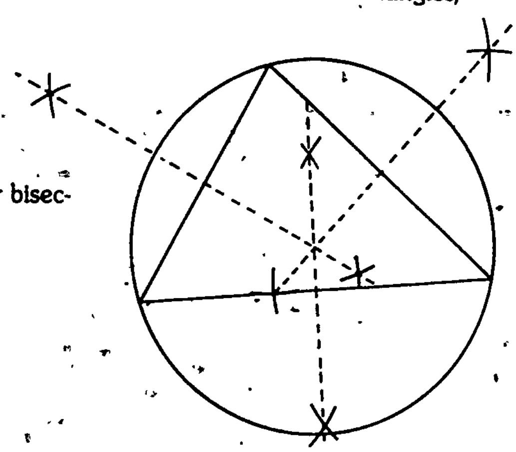
determine that the point of concurrency (called the circumcenter) is the center of a circle which circumscribes the triangle.

Materials

Worksheets with triangles of all types as drawn (or have students draw their own triangles)

Compass

Straightedge



Method

Have students construct the perpendicular bisector of the sides of a triangle.

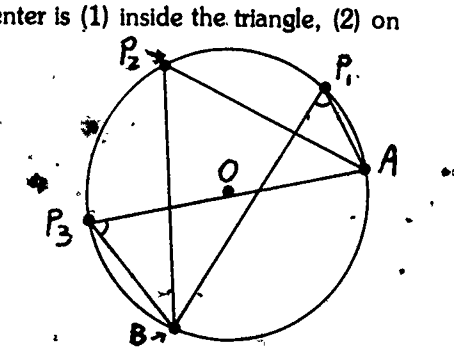
Draw the circle as shown in the diagram.

Repeat for all types of triangles — acute, obtuse, right, isosceles, equilateral.

Evaluation

Given a triangle, have students find the circumcenter and draw the circumscribed circle.

Ask students to draw triangles in which the circumcenter is (1) inside the triangle, (2) on the triangle and (3) outside the triangle.



Using the circle already drawn on the worksheet, have students select points on the circle. Using protractors have students determine that m$\measuredangle$APB = ½m (AB) for each point P selected.

Help students formulate the theorem:
The measure of an inscribed angle of a circle is one half the measure of its intercepted arc.

Return to the navigation problem. Suppose the captain wants to keep outside the arc of the circle. This means that the m$\measuredangle$ASB must be less than m$\measuredangle$APB. Why? Thus, as long as the captain maintains an $\measuredangle$ASB so that m$\measuredangle$ASB < M$\measuredangle$APB the ship is safe.

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Comment — This lesson can also lead to theorems about the measures of angles formed by secants or tangents to a circle. Also, the problem of finding the center of the circle which contains points A, B and P such that $m\angle APB = 60°$ can be considered.

Numerical exercises can be developed in which measures of the arc of a circle are given and the measures of the inscribed angles are to be found and vice versa.

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Sample Course Plan

Course title — *Geometry through Transformations (year)*

Course description

The course begins with a study of relations and functions. Geometry is considered as a continuation of the study of algebra. The concept of transformations is introduced early in the course. Transformations are used throughout the course to teach characteristics of geometric figures. Through the study of isometries and dilation, students will be provided useful applications of the function concept. However, the major purpose for introducing transformations in the study of geometry is to provide a useful tool for solving geometric problems.

Course objectives — The student will

1. classify examples of relations according to the properties; equivalence, order, function, isometry, similarity.
2. explain why transformations in the plane are relations.
3. identify basic geometric figures.
4. classify basic geometric figures.
5. use geometric relations to compare basic geometric figures.
6. use a compass and straightedge to construct various geometric figures.
7. use a mira to draw various geometric figures.
8. use transformations to describe properties of various geometric figures.
9. make some deductions after performing some geometric constructions according to given directions.
10. solve some geometric problems which involve numerical results.
11. find the area, volume and perimeter of standard geometric figures.
12. use transformations to show how the formulas for area of plane figures were derived.
13. apply measurement concepts to solve problems.
14. use transformations to prove theorems about congruence.
15. use the theorems about congruence to analyze geometric situations.
16. use the theorem about corresponding parts of congruent figures to analyze geometric situations.
17. apply the properties of similarity transformations to solve ratio and proportion problems in a geometric context.
18. use the distance formula.
19. apply the slope condition for perpendicularity and parallelism.
20. use coordinate geometry to establish some of the traditional resluts of Euclidean geometry.
21. use coordinate geometry to perform some transformations in the plane.

Course content

- Let's relate
  1. arrow diagrams
  2. domain/range
  3. ordered pairs
  4. mappings
  5. functions
  6. graphs
  7. transformations
     a. isometries
        (1) translations
        (2) rotations
        (3) reflections
        (4) glide reflections
     b. similarities
  8. topology
- Tradition
  1. points
  2. lines
     a. parallel
     b. perpendicular
     c. skew
     d. segments
     e. intersecting
     f. rays
  3. angles
     a. acute
     b. right
     c. obtuse
     d. straight
     e. supplementary
     f. complementary
  4. triangles
     a. equilateral
     b. isosceles
     c. scalene
     d. right
     e. 30-60-90
     f. 45-45-90
  5. quadrilaterals
     a. rectangles
     b. squares
     c. parallelograms
     d. trapezoids
     e. rhombii
     f. kires
  6. circles
     a. circumference
     b. radii
     c. diameter
     d. tangents
     e. chords

f. inscribed angles
g. central angles
h. congruence
i. similarity
j. area and perimeter

- Triangles and things
  1. mira constructions
  2. compass and straightedge construction
  3. symmetry
  4. transversal of parallel lines
  5. dimensions of plane figures
- Measurement
  1. area
  2. volume
  3. perimeter
  4. formulas via transformations
- Congruence
  1. congruence through isometries
  2. corresponding parts of congruent figures
  3. triangle congruence principles
- Similarities
  1. similarity transformations
  2. ratio and proportion
- Coordinate systems
  1. distance formula
  2. slope
  3. midpoints of segments
  4. perpendicular and parallel lines
  5. proofs using coordinates

Instructional activities

(The following, and many other types of activities, may be developed at the local level.)

Studying, working through and discussing basic content of geometry, e.g., Isosceles Triangle Theorem

Studying relationships among isometries, e.g., the product of two reflections of parallel lines is a translation

Relating geometry to the real world, e.g., identifying objects with symmetry, consider real world problems — applications. (See Sample Activity *Reflections — Geometry through Transformations)*

Constructing and drawing geometric figures which require the use of various geometric tools and methods, e.g., paper folding, compass, straightedge and mira, thus providing students opportunities for original investigations. (Problems should be posed for students to make deductions after having performed geometric constructions according to specified directions.)

Evaluation

Teacher-made test items or similar means of evaluating student performance

Evaluation of problems solved which are real-world applications

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Evaluation of constructions and drawings and deductions based on findings from performing constructions

Evaluation of individual or group projects

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Geometry through Transformations
Sample Activity

Activity title — *Reflections*

Time

One or two class periods

Activity objective

The student will discover that the composition of two reflections about parallel lines is equal to a translation of twice the distance between the lines.

Materials

Worksheet with parallel lines, mira or tracing paper (the parallel lines should be at various distances apart and with different orientations).

Method

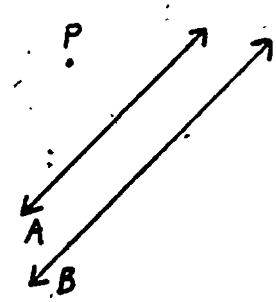
Reflect P about A then B. Have students compare PP with the distance between A and B.

Repeat using lines C and D. Reflect P about C first then D.

Repeat for lines E and F.

Repeat for lines G and H (as an alternative, students could reflect a segment or a triangle rather than a point).

Help students discover the desired theorem.



Evaluation

Students should be given two parallel lines and asked to describe an equivalent transformation to the composition of the two reflections.

Students could be given a translation and be asked to represent the translation as the composition of two reflections.

Students can be asked to find image points, given points and two parallel lines of reflection.

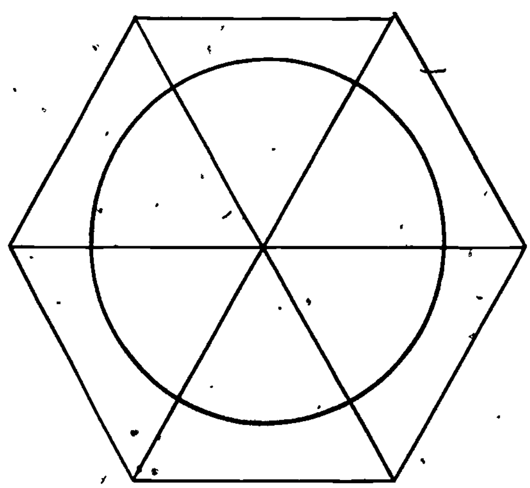
Note that a similar activity could be devised for the composition of two reflections about intersecting lines.

The Minimum Bisector

What is the curve of minimum length which bisects the area of an equilateral triangle?

Solution

Reflect the triangle and the bisecting curve repeatedly, keeping one vertex fixed, and thus produce a regular hexagon. The closed curve cuts the area of a hexagon in half, so has a fixed area inside. Consequently, if its perimeter is a minimum the curve is a circle with center at the fixed vertex.



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Sample Course Plan

Course title — *Basic Principles of Geometry (quarter)*

Course description

This course will present the basic ideas of geometry including the nature of angles, triangles, congruence, geometric inequalities, perpendiculars and parallels. The course will be taught using postulates and theorems in an effort to teach the nature of direct and indirect proof.

Course objectives — The student will

1. determine from a list which terms will be undefined in geometry.
2. describe points, lines, planes and spaces in set terminology.
3. recognize and define direct and indirect proof.
4. define collinear and coplanar.
5. define and illustrate convex sets.
6. define, illustrate and measure angles.
7. define and identify complementary, supplementary, right, congruent and vertical angles.
8. work simple problems with angles.
9. define and illustrate an angle's bisector.
10. define and illustrate the various types of triangles.
11. state the three basic congruence postulates.
12. prove problems using the three basic congruence postulates.
13. define, illustrate and work problems concerning medians and altitudes.
14. define, illustrate and work problems concerning exterior angles of a triangle.
15. state theorems concerning inequalities within a triangle and between two triangles.
16. identify and describe perpendiculars within planes and perpendiculars found by planes and lines.
17. define and describe parallel lines and planes.
18. prove theorems concerning the five main types of quadrilaterals.

Course content

Defined versus undefined terms

Nature of proof

Angles

Triangles

Inequalities

Perpendicular lines

Parallel lines

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Instructional activities

(The following activities and many others may be developed at the local level.)

- Play geometry vocabulary games.
- Activities on Tessellations and Dissections. (See Sample Activity *Geometry — Tessellation)*
- Illustrate congruences, inequalities, perpendiculars and parallels with construction paper and paper clip models.
- Let students present proofs to class.

Evaluation

- Teacher-made test
- Class presentation evaluation
- Individual or group project evaluation

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Geometry
Sample Activity

Activity title — *Tesselations*

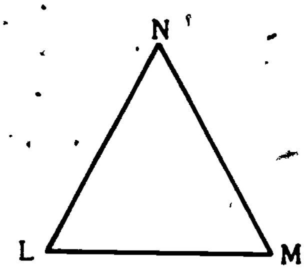
Activity objective

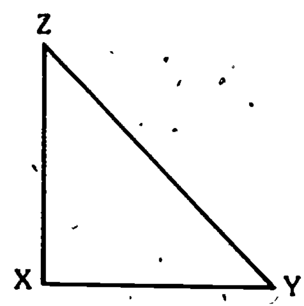
The student will demonstrate the relationship between the triangle and rhombus, trapezoid, parallelogram or hexagon.

Time — One class period

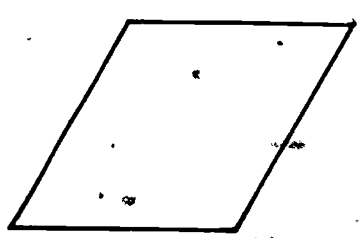
Materials

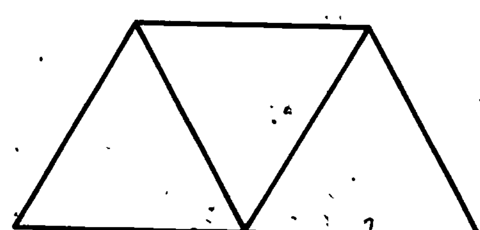
. Cutouts of triangles LMN and XYZ

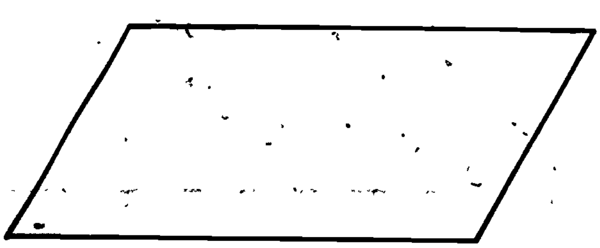


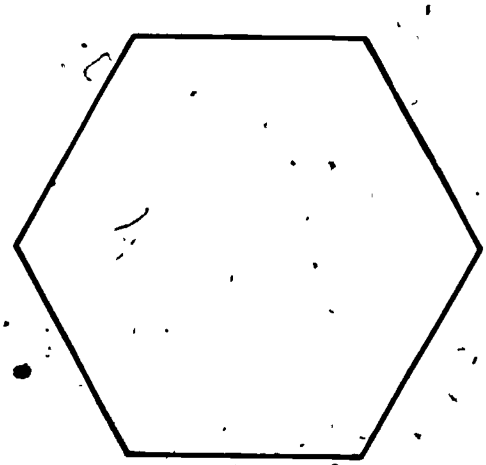


Patterns for the figures below.









Method

Introduce "tessellation" as a Latin word which means to cover with tiles.

Using $\triangle LMN$, determine if the rhombus can be covered. If so, sketch the solution.

How do you make the trapezoid? Sketch your solution.

How do you make the parallelogram and hexagon? Sketch your solutions.

Can you make a large equilateral triangle using exactly four of the small tiles? Sketch your results.

Take a closer look at the hexagon. Can you find another shape within this region? Sketch it on paper.

Using tiles shaped like $\triangle XYZ$, can you make a square using exactly four tiles? Can you make a larger square using exactly eight triangles? Sketch your results.

Can you make a larger right triangle using exactly four tiles of $\triangle XYZ$?

Evaluation

Evaluate the sketches on the basis of correctness.

Creating tesselations should be encouraged; teacher or peer evaluation of the tessellations could be used.

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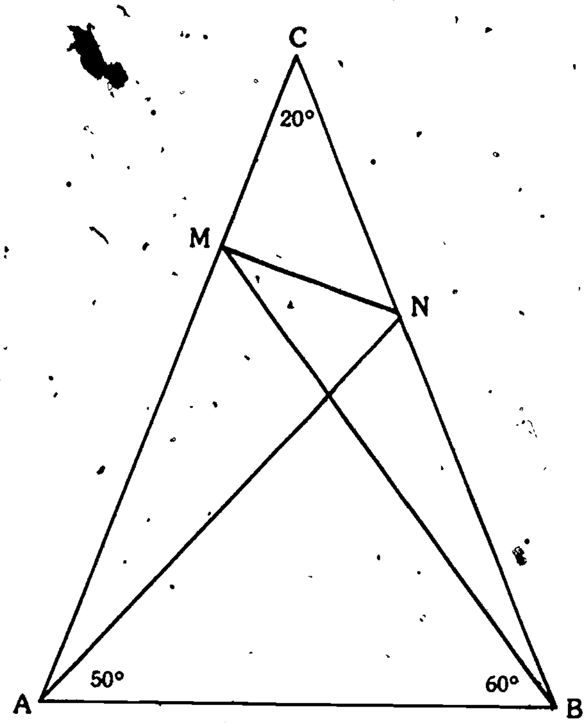
The Elusive Angle

An isosceles triangle ABC has a vertex angle C = 20°. Points M and N are so taken on AC and BC that angle ABM = 60°, and angle BAN = 50°. Without resorting to trigonometry, prove that angle BMN = 30°.

Solution

Since the sum of the angles of a triangle equals a straight angle or 180°, then ∡CBA = ∡CAB = 80°, ∡CBM = 20° and ∡BAN = 50° = ∡BNA (so BN = AB). Draw MR parallel to AB and AR intersecting BM at D. Draw ND. By symmetry, triangles ABD and MDR are isosceles and hence equiangular. Then BD = AB = BN, so ∡BND = ∡BDN = 80°, and ∡NDR = 40°. Now ∡MRC = 80°, so ∡NRD = 40° = ∡NDR and ND = NR. Then since DM = MR, NM is the perpendicular bisector of DR. Therefore ∡BMN = 60°/2 or 30°.



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Sample Course Plan

Course title — *Computer Science I (quarter)*

Course description

This course is designed to introduce the student to the history of electronic computing, elementary vocabulary and career opportunities in computer science.

Course objectives

The student will

1. relate the general history of electronic computing.
2. be conversant in the vocabulary of computer science.
3. be aware of career opportunities in the computer industry.
4. be aware of the various sizes and capabilities of computers.
5. state the names of several computer manufacturers — from large scale computers to personal computers.
6. construct a set of instructions for others to follow in accomplishing a task.
7. follow a set of instructions developed by others.

Course content

History of electronic computing

Overview of current technology

Vocabulary of computers

Careers in computer science

Algorithms — construction and interpretations

Instructional activities

The following activities and many others may be developed at the local level.

Working selected activities in a suitable textbook

Using employment advertisements in newspapers to determine nature of careers in computer science

Writing algorithms (in ordinary sentences) for other students to follow

Interpreting algorithms that others have written (See Sample Activity *Human Flow Chart — Computer Science I*)

Visiting computer installation or have a small computer brought to the school for a demonstration

Evaluation

Teacher-constructed tests

Peer evaluation of algorithms

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Computer Science I
Sample Activity

Activity title — *Human Flow Chart*

Activity objective

The student will follow a flow chart developed by others.

Time

Approximately 45 - 60 minutes, including class discussion time, may be needed for this activity.

Materials

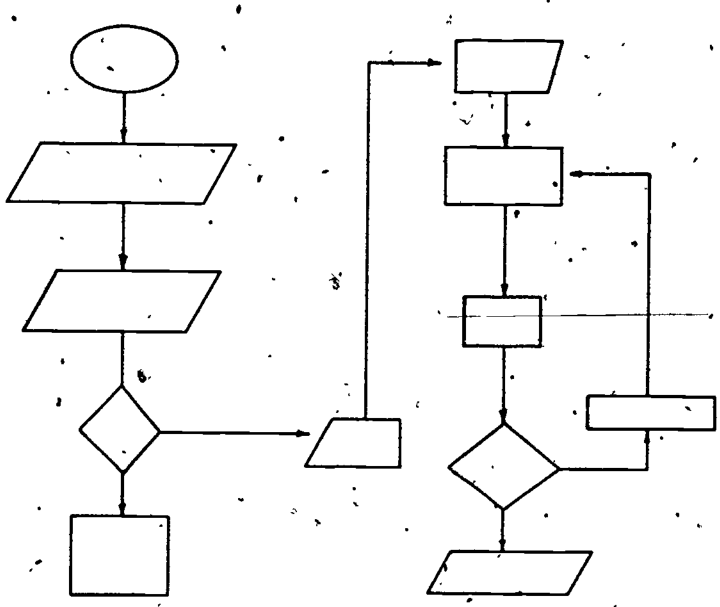
Masking tape
Poster board
Marking pens
Open floor space (push chairs toward the wall in a classroom)

Method

Place flow chart symbols on the floor, with tape connecting them to show flow directions. Include a *choice branch* or two. Allow several students to be the computer, moving along the flow chart, computing values of variables along the way. At output times they may direct a printer (other student) to display data as directed by the flow chart. At input times the computer can receive data from the operator (rest of the class). As variable vlaues change, the computer can direct an arithmetic control unit (student at the blackboard) to record changes.

Evaluation

Similar flow charts may be included on subsequent tests with interpretation required of the students.



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Sample Course Plan

Course title — *Computer Science II (quarter)*

Course description

This course is designed to provide students with an overview of the present state of the art in computing, programming concepts and introduction to a programming language.

Course objectives

The student will

1. be aware of current trends in the computer industry.
2. construct and interpret algorithms in both mathematical and nonmathematical settings.
3. interpret number names and symbols used in computer related literature.
4. perform arithmetic operations peculiar to computer related problems.
5. illustrate algorithms with flow charts.
6. interpret flow charts constructed by others.
7. translate a flow chart into a program in a higher-level computer language.
8. read and follow an instruction manual for programming a computer or operating a computer or computer terminal.

Course content

Survey of current trends and foreseeable advances in future technology

Use of algorithms to obtain problem solutions

Flow charts — construction and interpretation

Special uses of relational symbols: +,

Hierarchy of operations and use of parentheses

Application of F(x) notations, functions and arguments

Numeric and alpha numeric variables

Logarithms and exponents

Branching in computer programs

Introduction to a higher-level computer language, e.g., Basic, APL, Fortran, PL/1 Pascal (See Sample Activity *Comparison of Languages — Computer Science II*)

Instructional activities

(The following activities and many others may be developed at the local level.)

Working selected activities in a suitable textbook

Constructing flow charts that model algorithms

Stating algorithms that compute solutions to problems

Writing programs to be run on a computer

Preparing a short research paper on a specialized area of computer science or data processing.

Preparing a lesson on a suitable data processing topic (such as word processing) for presentation to a business education class

Hand-checking flow chart and programs to determine correctness of logic and language elements

Evaluation

Teacher-constructed tests

Evaluation of research papers

Peer evaluation of algorithms and flow charts

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Activity title — *Comparison of Languages*

Activity objective

The student will interpret number names and symbols used in computer-related literature.

Time

1 week (near the end of the quarter)

Method

Prepare a program in the language most common to the students. Give every student a copy. Assign individuals (or groups) a new language as the medium for writing the same program.

Materials

Resource books, periodicals and other available sources

Evaluation

Language elements from several languages may be included on subsequent tests. If a computer is available for the languages used, successful runs may be used as evaluations.

Sample Course Plan

Course title — *Computer Science III (quarter)*

Course description

This course is designed to provide students with exposure to various computer languages, advanced programming expertise and experience in applying computers to assist in problem solving.

Course objectives

The student will

1. state the names of several computer languages and the general differences among them.
2. describe the nature of storage files and illustrate the use of files by flow chart or program.
3. compute in bases 2, 8, 16 and convert among bases 2, 8, 10 and 16.
4. give a brief description of an operating system and its function.
5. give a brief synopsis of the current state of the art regarding peripherals such as magnetic disks, magnetic tapes and printers.
6. write a program to solve an assigned problem and assure that the program's logic is accurate and well-documented.

Course content

- Comparison of high-level computer language
- Introduction to machine-level programming
- Introduction to microprocessors
- Binary, actual, hexadecimal arithmetic
- Computer graphics
- Computer peripherals
- Computer operating system program
- Introduction to computer file structures
- Software systems analysis
- Applications programming

Instructional activities

(The following activities and many others may be developed at the local level.)

- Work selected activities in a suitable textbook
- Arranging for a demonstration (or extended loan use) of a microcomputer with graphics capability
- Arranging for a visiting speaker on printers (types, costs, how they are interfaced with computers) (other peripherals such as magnetic tape drives are also appropriate)
- Assigning a class project in which different individuals or groups must produce subroutines programmed to fit into a larger main program; this activity is an introduction to systems pro-

gramming (See Sample Activity *A Calculator Mode for the Computer — Computer Science III*)

- Programming a system for use at the school, such as handling yearbook monies or automating the library's overdue book reminders

Evaluation

- Teacher-constructed tests
- Successful runs of programs on a computer
- Successful operation of a computer (when available) from start up to shutdown

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Activity title — *A Calculator Mode for the Computer*

Activity objective

The student will identify and describe subroutines.

Time

1 week

Method

On the first day, assign each student (or group) a portion of a calculator mode for a computer, such as square root, y^x, factorial, ordinary arithmetic operations, trigonometric functions and logarithms. Also assign line number ranges for each subroutine to assure meshing of all program parts. During the first three days present class lecture/discussions in which the main body of a calculator mode is programmed — accepting input from the terminal, processing that input to decide which subroutine will be useful, printing of the answer when it is returned from the subroutine.

During the fourth and fifth days, spend some time helping students over the rough spots in their programming. Throughout the exercise, stress documentation techniques — both within the program itself and external to the program in paragraph narratives submitted by the student.

Note — Your computer may already have a calculator mode; however, this activity is a productive exercise.

Materials

Computer, hand-held calculator

Evaluation

Systematic testing of the various subroutines will provide evaluation.

Sample Course Plan

Course title — *Elementary Algebra (year)*

(This plan may be partitioned into segments for semester or quarterly organization of courses.)

Course description

This course is designed to introduce the student to the arithmetic of polynomials, solutions of various forms of equations, techniques of graphing on the line and in the plane, properties of rational and irrational numbers and applications of algebra to problem situations.

Course objectives

The student will

1. classify the real numbers into subsets such as natural whole, integers, rational and irrational.
2. correctly remove symbols of grouping from an expression containing numbers, variables or a mixture of both and containing a mixture of operations.
3. determine the correct order of operations to be applied in an expression having no symbols of grouping.
4. add, subtract, multiply and divide monomials and apply laws of exponents with integers.
5. substitute and evaluate variable expressions.
6. translate problem situation into mathematical symbols and translate mathematical expressions into words.
7. state and illustrate the axioms of the real number system.
8. identify and apply the following properties — closure, commutative, associative, distributive, identity and inverse.
9. add and subtract combinations of signed numbers and signed variable expressions.
10. multiply and divide combinations of signed numbers and signed variable expressions.
11. raise signed numbers to powers.
12. substitute and evaluate variable expressions involving absolute value.
13. identify and apply properties of equality or inequality.
14. solve linear equations with one variable and with more than one variable.
15. solve linear equations with decimal fractions or coefficients.
16. use a common denominator to change an equation with common fractions as coefficients to an equation with integral coefficients.
17. solve linear inequalities.
18. graph solutions of linear inequalities.
19. break down a composite whole number (of reasonable size).

[illegible] find the product of a monomial and a polynomial.

21. factor a composite polynomial into a product of a monomial and polynomial.
22. find the product of two binomials.

23. factor a composite trinomial into the product of two binomials.

24. factor by inspection certain special cases such as the difference of two squares or the difference of two cubes;

25. construct a graph of a linear function from a table or ordered pairs.

26. determine the slope and y-intercept of a linear equation.

27. construct a graph of a linear function using slope and y-intercept.

28. transform linear equations into slope-intercept form.

29. determine a point of intersection for two linear functions by graphing.

30. determine a point of intersection (simultaneous solution) for two linear functions by solving equations.

31. compute approximations to irrational numbers with and without the aid of a calculator or computer.

32. distinguish between rational numbers and irrational numbers using definitions based on fractions and decimal values.

33. perform arithmetic operations on radical forms with like and unlike indexes.

34. change a fraction with a radical form in the denominator.

35. solve equations of the form $K = x^n$ for fixed k and integer n.

36. apply rules of exponent manipulation using numbers with whole numbers exponents.

37. apply rules of exponent manipulation using numbers with negative exponents.

38. apply content to real world problems throughout the course.

Course content

Symbols of grouping

Order of operations

The arithmetic of polynomials

Evaluation of variable expressions by substitutions

Translation of variable expressions by substitutions
1. Nouns - variables
2. Verbs - operations and relational symbols
3. Sentences - equations

Formal properties of the real number system

Signed number operations with variable expressions

The absolute value function

Solving linear equations

Solving linear inequalities and graphing solutions

Factorization of composite whole numbers

Multiplication of polynomials using the distributive property
1. Product of a monomial and a polynomial
2. Product of two binomials

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Factorization of polynomials
1. Common monomial factors in all terms
2. Products of binomials
3. Special cases

Irrational numbers

Linear equations with two variables
1. Graphs
2. Finding intercepts
3. Slope
4. Slope-intercept form
5. Simultaneous solution of two linear equations
   a. graphing
   b. computation

Linear inequalities

Numbers with square root radicals
1. Finding equivalent forms
2. Arithmetic operations
3. Rationalizing denominators
4. Simple equations

Radicals and polynomials

Rules for exponents
1. Whole number exponents
2. Negative exponents

Quadratic equations

Algebraic fractions

Equations with algebraic fractions

Applications in problem situations

Instructional activities

(The following activities and many others may be developed at the local level.)

General classroom presentations and discussions

Individual assignments

Working in small groups

Peer tutoring or one-to-one help during class sessions.

Contracting and other forms of self-paced instruction for accelerated students

Assessing prior to course and prescribing instruction

Simulation exercises such as factoring bees (See Sample Activity *Factoring Bee — Elementary Algebra*)

Multitext library

Researching topics, e.g., biographical sketches

Evaluation

Frequent short progress checks

Unit tests

Homework checks

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Elementary Algebra
Sample Activity

Activity title — *Factoring Bee*

Activity objective

The student will translate an algebraic expression into a product of its factors.

Time

One or two class periods may be scheduled for this activity.

Materials

Chalk and chalkboard or other scorekeeping devices

Timing device

Method

Partition the class into two teams, roughly equivalent in ability.

Ask each student to make up a factoring problem (and the answer).

Take turns as follows:

Player 1 on team A gives a question to player 1 on team B. If player 1-B gets it right, team B scores 1 point. If player 1-B gets the problem wrong, team A can score a point if player 2-A can successfully answer the question. After this pattern, play continues. Player 1-B asks 1-A; then player 2-A asks 2-B.

Evaluation

Strengths and weakness will surface in the contest itself usually. Hence the activity itself serves as an evaluation.

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Elementary Algebra Problem for Students

A Student Approach to a Rowing Problem

The problem is, "John rows up the river at 6 mph and down the river at 8 mph. What would John's rate be in still water and what is the rate of the current of the river?" A student says, "To solve this, I add 6 mph and 8 mph and divide by 2. To find the rate of the current, I subtract 7 from 8 or else subtract 6 from 7. They both give the same answer." Is the student correct?

Solution

Yes, the student is correct. Solving this problem using a standard method, one gets the following.

x = rate of boat in still water
r = rate of current

$$\begin{array}{r} x + r = 8 \\ \underline{x - r = 6} \\ 2x = 14 \\ x = 7 \end{array}$$

Thus $r = 1$

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Sample Course Plan

Course title — *Intermediate Algebra (year)*

(This plan may be partitioned into segments for semester or quarter organization of courses)

Course description

Algebra is designed to include topics from the algebra of real numbers, the algebra of the plane, trigonometry, and enrichment topics that develop the student's overall mathematical maturity.

Instructional objectives

The student will

1. manipulate exponential forms of numbers, including fractional exponents.
2. convert an equation of the form $y = b^x$ to a logarithm equivalent form, $\log_b y = x$ and vice versa.
3. compute products, quotients and powers using the rules of logarithms in base 10.
4. solve equations that involve exponents and logarithms.
5. sketch the graph of a given member of the family of curves $y = b^x$ for $b > 0$.
6. sketch the graph of a given member of $y = \log_b x$ for $b > 0$ and $x > 0$.
7. use a hand-held calculator to compute x^y, log x and 1nx.
8. give an explanation of why $\log_b 1 = 0$ for any $b \neq 0$.
9. write equations for lines given various combinations of required information (such as slope, points on the line).
10. define a parabola.
11. give standard form for a given quadratic function.
12. plot a graph of a parabola both with axis parallel to the y-axis and parallel to the x-axis, with vertex either on or off of the origin.
13. find the directrix and focus for a given quadratic function.
14. plot a graph of a circle with center either on the origin or off.
15. write the equation of a circle when given sufficient information.
16. find the center and radius of a circle when given the general equation $x^2 + ax + y^2 + by = c$.
17. sketch a hyperbola and its asymptotes.
18. sketch half planes when given linear inequalities.
19. determine planar regions which are graphs of inequalities, for example, $y < x^2 + 4x + 4$.
20. determine planar regions which are interiors or exteriors of circles, for example, $x^2 + y^2 > 4$.
21. find regions in the plane that satisfy a system of inequalities and equations.
22. find a feasibility region for systems of inequalities and equations that model a problem situation.

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23. find zeros or roots of polynomials by graphing for linear functions and quadratic functions of one variable.

24. find roots of polynomials by factoring.

25. perform division of polynomials by long division and by synthetic division.

26. solve quadratic equations by factoring, completing the square and using the quadratic formula.

27. derive the quadratic formula.

28. graph solutions to quadratic equations.

29. apply the theory of quadratic equations to problems of motion and maximization.

30. find the equation of the parabola passing through three given points when appropriate.

31. find complex roots of a quadratic function whose graph misses the x-axis.

32. decide for a given quadratic function whether it misses, touches or crosses the x-axis.

33. convert complex numbers from one form to another among the forms (a,b), $a + bi$ and $a + \sqrt{-b^2}$.

34. perform addition, subtraction, multiplication and division on complex numbers.

35. graph i, i^2, i^3, i^4.

36. solve systems of 2 or 3 linear equations by algebraic manipulations.

37. solve systems of 2 or 3 linear equations by use of determinants and Cramer's Rule.

38. use properties of determinant to simplify determinant computations.

39. solve systems of linear and quadratic functions.

40. solve systems of lines and circles.

41. use common trigonometric functions with both angular and radian measure.

42. compute domain and range values of the wrapping function.

43. graph variations of sine, cosine and tangent functions with different values of frequency, amplitude and period where appropriate.

44. describe appropriate physical situations with trigonometric functions.

45. determine if an equation involving trigonometric functions is an identity.

46. plot inverses of trigonometric functions with appropriate domain restrictions.

Course content

Rules of exponents for fractional exponents

Logarithms in base 10

Analytic geometry
1. Lines
2. Parabolas
3. Circles
4. Hyperbolas

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Inequalities in the plane
1. Half-planes
2. Inequality regions
3. Interiors of circles
4. Introduction to linear programming

Roots of polynomials
1. Factoring
2. Synthetic division
3. Quadratic equations
4. Complex numbers

Systems of equations
1. Linear systems
2. Determinants and Cramer's Rule
3. Systems of lines and parabolas
4. Systems of lines and circles

Circular functions
1. Definitions of trigonometric functions
2. Angular measure and radian measure
3. The wrapping function
4. Graphing techniques - frequency, amplitude, period
5. Trigonometric identities
6. Inverse trigonometric functions

Instructional activities

The following activities and many others may be developed at the local level.

Having general classroom presentations and discussions

Working in small groups

Peer tutoring or one-to-one help

Contracting and other forms of self-paced instruction for accelerated students

Assessing prior to course and prescribing instruction

Using multitext library

Researching topics, e.g., computer applications, modeling (See Sample Activity *Applications of Parabolas — Intermediate Algebra*)

Evaluation

Frequent short progress checks

Unit tests

Homework checks

Intermediate Algebra Sample Activity

Course title — *Applications of Parabolas*

Activity objective

The student will identify, site and describe applications of parabolas.

Time.

One to two weeks may be allowed for research papers and two or three days for class presentation.

Method

Assign research topics on applications of parabolas, such as headlight reflectors, flash reflectors, telescope reflectors, radar antennas, microphones, projectile motion.

Materials

Print and nonprint materials for research.

Evaluation

Questions about these reports may be included on subsequent reports.

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Intermediate Algebra Problem for Students

Antifreeze Problem

A 21-quart-capacity car radiator is filled with an 18 percent alcohol solution. How many quarts must be drained and then replaced by a 90 percent alcohol solution for the resulting solution to contain 42 percent alcohol?

Solution

One quart of the old solution differs from one quart of the new (or average) solution by -24 percent; while one quart of the solution to be added differs from the new solution by $+48$ percent. Hence, there must be two quarts of the old solution for each quart of the added solution. So $1/3$ of the original radiator content or 7 quarts must be drained.

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Sample Course Plan

Course title — *Algebra of Matrices (quarter)*

Course description

This course is designed as a supplementary course for enrichment beyond the usual course in the algebra of real numbers. Matrices provide an algebraic structure with properties different from real number properties. The course includes operations with matrices and applications of matrices in the algebra of real numbers.

Course objectives

The student will

1. state the definition of a matrix.
2. state the order of a given matrix.
3. give an example of a matrix with a stated order.
4. add two matrices which are compatible.
5. find the product of a scalar number and a matrix.
6. determine if two matrices are compatible for multiplication.
7. multiply two matrices which are compatible.
8. list and illustrate the various algebraic properties of matrices.
9. give the additive inverse for a matrix.
10. give the multiplicative inverse for a matrix, when appropriate.
11. determine the transpose of a given matrix.
12. find determinants for 2×2 and 3×3 matrices.
13. find the product of a matrix and a column vector of variables and make application to a set of linear equations.
14. apply Cramer's Rule to solve systems of linear equations in the 2×2 and 3×3 cases.

Course content

Definition and examples of matrices

Addition of matrices

Multiplication of a matrix by a scalar number

Multiplication of two matrices

Algebraic properties of matrices

Additive and multiplicative identities

Additive and multiplicative inverses

Transpose of a matrix

Determinants of matrices

Matrices and linear equations

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Cramer's Rule

Powers of matrices

Instructional activities

The following activities and many others may be developed at the local level.

Having general class presentations and discussions

Working in groups and as individuals. (See Sample Activity *Matrices and Network Theory — Algebra*).

Researching mathematical topics including applications

Peer tutoring

Writing step-by-step algorithms for matrix calculations

Writing computer programs that incorporate matrices.

Evaluation

Frequent progress checks throughout this course

*Unit tests throughout the course

Evaluation of presentation of research reports

Completion of homework assignments may be monitored as a means of checking ongoing progress

Algebra of Matrices
Sample Activity

Activity title — *Matrices and Network Theory*

Activity objective

The student will create matrices and networks for situations presented.

Time

Two or three class periods may be scheduled for this activity.

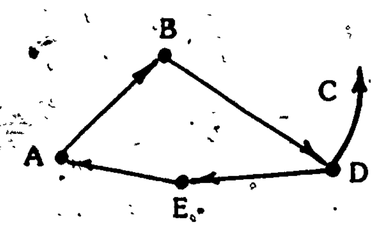
Method

The teacher should give a brief introduction to *transportation* networks via lecture/discussion or other suitable technique. Illustrations should include construction of adjacency matrices. For a network of n points $p_1, p_2, \ldots p_n$, construct a square $n \times n$ matrix M defined by $M_{a,b} = 1$ if point B is adjacent to point A and $M_{a,b} = 0$ otherwise. Leave an unsolved problem or two for homework. Let students present their solutions during the next day or two and pose problems of their own.

Materials

Charts with networks and associated adjacency matrix

Example



network

	A	B	C	D	E
A	0	1	0	0	0
B	0	0	0	1	0
C	0	0	0	0	0
D	0	0	0	0	1
E	1	0	0	0	0

adjacency matrix

Evaluation

On subsequent tests, students may be asked to provide adjacency matrices for given networks. Also, students may be asked to create both networks and matrices.

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Sample Course Plan

Course title — *Probability and Statistics (quarter)*

Course description

This course is designed to provide an introduction to descriptive and inferential statistics. The course includes measures of central tendency, dispersion and correlation for bivariate data. Elementary probability is used to explain binomial and normal probability distributions.

Course objectives

The student will

1. construct common graphical data displays — dot array, scatter diagram, histogram.
2. compute mean, median, mode, range, percentiles, standard deviations and other common descriptive statistics for univariate data.
3. compute a correlation coefficient for a set of bivariate data.
4. distinguish between positive and negative correlation.
5. distinguish between correlation and cause and effect.
6. apply a linear regression equation in a practical setting.
7. compute compound probabilities for simple cases when events are independent.
8. interpret a histogram for a discrete probability distribution in which area is equated with probability.
9. read and interpret a table of binomial probabilities.
10. calculate simple binomial probabilities.
11. compute a standardized z-score for a normal distribution.
12. read and interpret a table of normal probabilities.
13. distinguish between problem situations which require binomial probability models and those which require normal models.
14. distinguish between descriptive and inferential statistics.
15. make simple inferences based on probabilities associated with binomial or normal distributions.
16. distinguish between a sample and a population.

Course content

Basic terminology

Measures of central tendency

Measures of dispersions

Measures of rank

Techniques of sampling

Graphical displays of data

Correlation

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Linear regression

Elementary probability

Discrete probability distributions

Binomial probability distributions

Normal probability distributions

Statistical inference

Instructional activities

(The following activities and many others may be developed at the local level.)

Having lecture/discussion (See Sample Activity *Probability and Genetics — Probability*)

Having experiments in class
1. ESP experiments
2. Product preference tests
3. Probability experiments

Working on group projects

Researching mathematical topics including applications

Studying collections of media use of statistics

Inviting guest speakers

Evaluation

Periodic teacher-made tests for most content

Evaluating group and individual projects

Tournaments involving teams or individuals by using questions presented on overhead projector displays so that graphs may be used.

Probability Sample Activity

Sample activity — *Probability and Genetics*

Activity objective

The student will apply probability to problem situations in genetics.

Time

One to three days may be scheduled for this activity.

Method

Arrange to swap class assignments with a biology teacher so the biology teacher can instruct your probability class in introductory genetics. According to the level of your class, discussion may be restricted to the $(P + Q)^2$ case or extended to higher levels of the binomial expansion.

Materials

Supplementary textual material

Evaluation

Problem assignments may be made at the time of instruction, or questions about the material may be included on subsequent tests.

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Sample Course Plan

Course title — *Introductory Analysis (year)*

Course description

This course is designed to expose the advanced student to a broad range of mathematical topics at a level which allows preparation for careers and enrollment in college programs which require mathematical maturity. Included are topics from real and complex analysis, set theory, logic, algebra, analytic geometry and trigonometry.

Course objectives

The student will

1. identify characteristics of subsets of complex numbers in relation to such properties as closure, order, density and completeness.
2. construct and analyze proofs in set theory.
3. exhibit a knowledge of the definition, notation and pictorial representations of set theoretic concepts.
4. determine the following, given a rule for an operation and a set on which the operation is defined.
   - an operational table
   - identity and inverse elements
   - the existence of closure
   - the existence of properties such as commutative, associative, and distributive
5. apply definitions to check for algebraic constructs such as group, field and finite geometrics — given an operational table, set of numbers or sets of points.
6. compute limits of sequences.
7. write a given finite series in a sigma notation, and write the finite series given a sigma notation.
8. determine convergence or divergence for infinite series using common tests and compute infinite sums where appropriate.
9. use mathematical induction in proving theorems.
10. use definitions or divisibility, and apply the Fundamental Theorem of Arithmetic in proving theorems.
11. identify and/or define
    - relation,
    - domain,
    - range,
    - function,
    - equivalence relations,
    - composition of function,
    - inverse function.
12. identify and/or define the following functions:
    - identity
    - constant
    - absolute value

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greatest integer

piece-wise defined; e.g., $F(x) = \begin{cases} g(x), \text{ if } x < a \\ u(x), \; a \leq x \leq b \\ v(x), \text{ if } x < b. \end{cases}$

13. describe a mapping such that
    the mapping is a function without an inverse,
    the mapping is a function with an inverse.

14. determine the following for a mapping or mappings
    the image of a given domain value,
    the preimage of a given range value,
    the inverse of the mapping,
    the composites of the mappings.

15. solve and graph
    linear and quadratic equations,
    linear and quadratic inequalities,
    absolute values (equalities and inequalities).

16. apply the concepts of relations and functions to the following.
    inverse and direct variation,
    vertical line test for f and horizontal line test for f'.

17. discuss given functions in terms of

symmetry	maximum
continuity	minimum
asymptotes	boundedness
slope	ultimate direction
rates of change	exclusions from the domain
intercepts	intervals in which zeroes occur.

18. describe geometric transformations in terms of functions.

19. identify some relations in a Cartesian product $A \times A$ as partitions of A.

20. transform into standard form and sketch the graphs of equations of conics.

21. derive an equation given its roots or derive a function given points of its graph.

22. find the points of intersection of conics with straight lines and conics with conics such as
    straight lines and parabolas,
    straight lines and circles,
    two parabolas,
    two circles,
    parabolas and circles.

23. state and use the following theorems.
    Rational Zeroes Theorem,
    Factor Theorem,
    Remainder Theorem,
    Fundamental Theorem of Algebra.

24. demonstrate the relationship between exponential (base 10 and e) and logarithmic functions through
    defining,
    translating their equations,
    graphing.

25. use laws of logarithms to solve equations.

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26. define trigonometric definitions, identities and laws to problem solving situations.

27. apply the Binomial Theorem to appropriate situations.

28. perform basic operations with vectors and apply these to problem situations.

29. perform basic computations involving complex numbers; solve equations having complex roots and graph complex numbers.

30. solve and graph equations written in polar form.

31. construct indirect proof.

32. determine equivalence between sentences involving conjunctions, disjunctions, negations and conditionals.

33. determine the truth tables for sentences involving the connectives of conjunctions, disjunctions, negations, conditional, biconditional and combinations thereof.

34. use Venn Diagrams to illustrate the relationships represented by the truth tables of the preceding objective above.

35. represent sentences in symbolic form and use the tools of truth tables and symbolic logic in assessing equivalence and validity.

Course content

Sets and symbolic logic
- Sets
- Union and intersection
- Subsets
- Truth tables
- Methods of proof
- Validity of arguments

Sequences and series
- Finite sequences and series
- Limits
- Infinite sequences and series
- Writing series (sigma notation)
- Mathematical induction

Ordered fields
- Field properties
- Direct proof
- Indirect proof
- Groups
- Rings
- Ordered properties

Algebra of vectors
- Number pairs and geometry
- Algebra of number pairs
- Parallel and perpendicular vectors
- Application of vectors

Analytic geometry
- Using numbers to describe points
- Algebraic properties of lines
- Quadratic equations and their graphs

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- Intersection of graphs
- Conic sections and their applications

Functions
- Relations
- Linear relation
- Functions
- The arithmetic of functions
- Polynomial functions
- The arithmetic of polynomials
- The factor theorem
- Rational roots
- Descartes Rule — Bounds
- Irrational roots of polynomials

The field of complex numbers
- Defining and representing complex numbers
- Addition of complex numbers
- Multiplication of complex numbers
- Fundamental theorem of algebra
- Relationships among roots and coefficients
- Polynomials with real coefficients

Graphs of polynomial functions
- Curve sketching
- Limits of a function
- Continuity
- Tangents to a curve
- Derivatives of polynomials
- Using derivatives in graphing
- Applications of maxima and minima

Exponential and logarithmic functions
- Exponential functions with rational exponents
- Exponential functions with real exponents
- The exponential function $(x,y) : y = e^x$
- Linear interpolation
- Composition of functions
- Inverse of functions
- Logarithmic functions
- Additional theorems and application
- Tangents to the graph of a $\log_b$ and $\exp_b$

Circular functions
- The unit circle
- The sine and cosine functions
- Graphing the sine and cosine functions
- Amplitude and period
- The tangent and cotangent functions and their graphs
- The secant and cosecant functions and their graphs
- Identities
- Angles and their measure
- The trigonometric functions.

Statements involving circular functions
- Sum difference and reduction formulas

- Double and half-angle identities
- Inverse values
- Inverse circular functions

Solving triangles

- The right triangle
- Law of cosines
- Law of sines
- Area of triangles
- Polar coordinates
- Polar graphs
- Powers and roots of complex numbers
- Complex numbers in polar form
- Multiplication of complex numbers
- DeMoivre's Theorem

Instructional activities

(The following activities and many others may be developed at the local level.)

Holding lecture/discussions

Having group projects

Researching mathematical topics including applications

Having oral presentations by students

Inviting guest speakers

Using multitext library

Using nonprint media

Using a resource library/laboratory

Using a programmable calculator or computer to assist in calculations

Participating in math tournaments.

Evaluating

Periodic teacher-made tests for content

Evaluation of group and individual projects

Evaluation of levels above mere knowledge and comprehension through problem-solving exercises in homework and on tests

Evaluation of oral presentations at the time of presentation for immediate improvement of deductive reasoning skills

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Analysis
Sample Activity

Activity title — *DeMorgan's Laws*

Activity objective

The student will verify DeMorgan's Laws.

Time

One or two days may be scheduled for this activity.

Method

Use sample sets and diagrams to verify DeMorgan's Laws for sets

$$A \cup B = A \cap B$$
and
$$A \cap B = A \cup B$$

use "set equality" arguments to prove both laws.

Compare these laws to their analogs in logic.

$$\sim(P \vee Q) = \sim P \wedge \sim Q$$
and
$$\sim(P \wedge Q) = \sim P \vee \sim Q$$

Materials

Resource Texts

Evaluation

Questions and subsequent tests may require students to illustrate or apply DeMorgan's Laws.

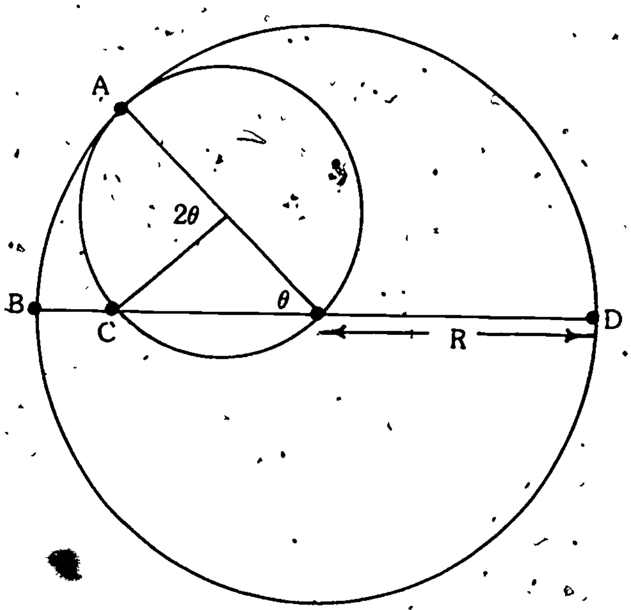
Introductory Analysis Problem for Students

The Hula Hoop

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the girl's waist.

Solution

Since motion is relative, consider the hoop as fixed and the girl whirling around. The original point of contact on the girl traverses the diameter of the hoop twice, and this is the required distance.



As the girl whirls, the original point of contact C on her waist traverses the diameter BD since arc AB = $R\theta$ = $(R/2)(2\theta)$ = arc AC.

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Introductory Analysis Problem for Students

The Lucky Prisoners

A jailer, carrying out the terms of a partial amnesty, unlocked every cell in the prison row. Next he locked every second cell. Then he turned the key in every third cell, locking those cells which were open and opening those cells which were locked. He continued this way, on the nth trip, turning the key in every nth cell. Those prisoners whose cells eventually remained open were allowed to go free. Who were the lucky ones?

Solution

The number of times T that the key was turned in the qth cell is equal to the number of divisors of q. Thus if $q = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ where the p's are distinct primes, then $t = (a_1 + 1)(a_2 + 1) \ldots (a_k + 1)$. Now if any a_i is odd, T is even and the corresponding cell eventually remained locked. If all the a_i are even, q is a square number, T is odd and the lucky occupants of the square cells found their cells eventually remained open.

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Sample Course Plan

Course title — *Calculus (year)*

(This plan may be partitioned into segments for semester and quarter organization of courses.)

Course description

This course includes a brief review of algebra; analytic geometry; limits and continuity; methods of differentiation and applications, exponential and logarithmic functions; methods of integration and applications, sequences and series, mathematical induction; volumes of revolution; area in the plane; work; arc length.

Course objectives

The student will

1. find solutions to equations and inequalities, factor and deal with exponential expressions.
2. locate points on the rectangular coordinate system and calculate the distance between two points in a plane, the distance between a point and a line and find the slope of a line.
3. write the equation of a line using the point-slope, slope-intercept and general forms.
4. identify families of curves.
5. evaluate limits and indicate where a function is continuous and where it is discontinuous.
6. find derivatives using limits, power rule, product rule, quotient rule and chain rule.
7. use derivative to solve for slope and to test for maximum or minimum.
8. determine the derivative of exponential functions (e^x and a^x) and logarithmic functions (1nx); identify 1nx and e^x as inverse.
9. evaluate antiderivatives (integrals) by inspection (predicting the answer and then checking by differentiation).
10. identify and use the Fundamental Theorem of Calculus to evaluate definite integrals.
11. use definite integrals to evaluate areas in the plane.
12. find antiderivatives by making elementary algebraic substitutions and using common functions — polynomials, exponentials, logarithms, powers of functions.
13. use sigma notation when given a sum; evaluate sums using sigma notation.
14. identify geometric sequences and find limits of sequences.
15. recognize and evaluate Riemann Sums by making partitions and summing areas of rectangles.
16. apply integrals to problems involving area in the plane, volumes of revolution, Hooke's law and work.
17. compute arc length, approximate areas in the plane using the trapezoidal rule.
18. find derivatives for all trigonometric functions and common combinations; evaluate inverse trigonometric functions with the use of right triangles.
19. compute derivatives of inverse trigonometric functions; use these derivatives in integration problems.

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20. use tables of integration formulas.

21. perform integration using the parts method, trigonometric substitution; integrate $p(x)/q(x)$ where p and q are polynomials.

Course content

Analytic geometry

Limits and continuity as applied to functions

The derivative
1. Formulas of differentiation
2. Graphing applications of derivatives
3. Trigonometric functions
4. Inverse trigonometric functions
5. Exponential and logarithmic functions
6. Differentiation using limits, power rule, product rule, quotient rule and chain rule
7. Finding slope, maximum and minimum

The integral (antiderivative)
1. Definite integrals
2. Applications
3. Fundamental Theorem of Calculus
4. Computation of areas in a plane
5. Tables of formulas
6. Integration by parts and trigonometric substitution

Sigma notation

Riemann sums

Trapezoidal rule

Instructional activities

The following activities and many others may be developed at the local level.

Assigning problems

Using film strips designed for a step-by-step approach to calculus

Explaining graphs, diagrams and model problems in detail using an overhead projector

Discussing weekly review sheets distributed as an overview of material presented

Lecturing with question and discussion periods

Identifying practical uses of the derivative, such as in maximum and minimum problems (See Sample Activity *Calculus Applications in Physics — Calculus*)

Discussing relationships between derivatives and antiderivatives

Discussing relationships between rectangular and polar coordinates

Sketching illustrations of problems involving areas in the plane and volumes of revolution

Evaluation

Scheduled teacher-made tests for most content areas

Evaluation of homework assignments at specific intervals to assess the progress of students between tests

Several pop quizzes between tests to determine whether or not students are keeping up with material presented and homework assignments.

Calculus Sample Activity

Activity title — *Calculus Applications in Physics*

Activity objective

The student will apply the calculus to problem situations.

Time

Two or three days may be scheduled for this activity.

Method

The topic of parabolas may be applied to projectile motion. The maximum height may be determined by techniques of differentiation and the force of gravity may be interpreted as acceleration in order to derive equations of motion.

$$a = -32 \rightarrow v = -32t + v_0 \rightarrow s = +16t^2 + v_0t + c$$

Materials

Physics resource texts

Evaluation

Calculus-based physics questions may be included on subsequent tests.

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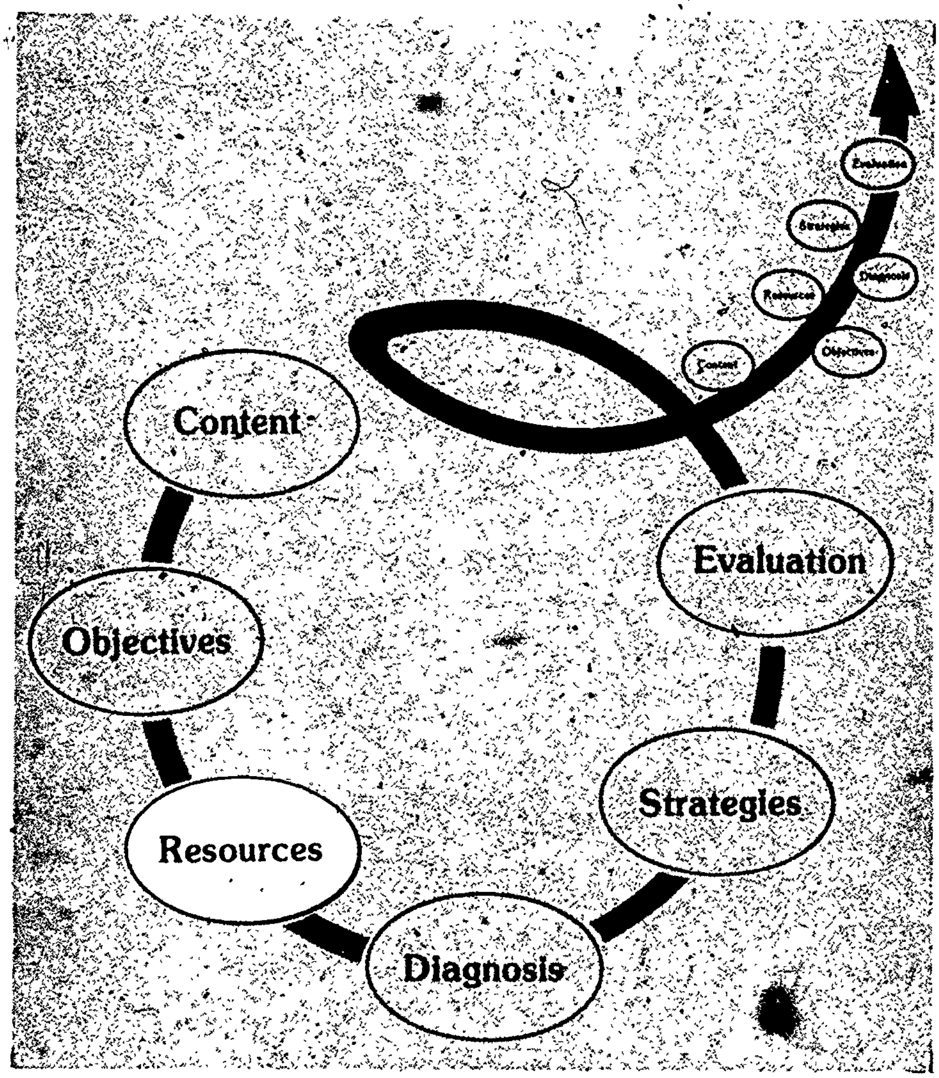
Content
Objectives
Resources
Diagnosis
Strategies
Evaluation

Instructional Resources to Support the Secondary School Mathematics Program

Educational media programs in Georgia public schools focus not only on the provision of instructional resources in all formats to support the curriculum but also on the use of those resources in addressing teaching strategies and learning activities to meet student needs in the most effective way. A combination of resources including print and nonprint materials, equipment essential for their use or production and programs, services and additional resources available through state, community and other educational agencies are necessary for effective support of instructional programs.

Innovative teachers, media specialists, administrators, curriculum specialists, students, board members and representatives of the community are cooperatively evolving a media concept that access to information in all formats and provision of services in production of locally designed learning materials. They can ensure effective use of appropriate materials to foster growth of listening, viewing, reading and inquiry skills. The Georgia Board of Education's Instructional Media and Equipment policy (IFA) requires that media committees composed of the groups mentioned above be involved in selecting materials and establishing procedures for effective use. Mathematics department chairpersons or content area representatives should contact their media specialists to become involved in or provide input into this planning process.

Ensuring access of teachers and students to information at time of need and preventing unnecessary duplication of resources will be accomplished when information about and location of all resources that support the mathematics program in a secondary school including simulation games, models, handheld calculators are available through the school's media center. In some systems, an additional resource service designed to augment the building media program is provided at system level for all schools. A community resources file, developed cooperatively by media and instructional staff, provides valuable information about local people, places, activities and unique resources to enhance the mathematics program.

Many professionally prepared, commercially published reviewing sources which are available in school media centers, system media collections, public and academic libraries are listed in

"Selected Sources of Information on Educational Media", available from Media Field Services, Division of Educational Media Services, Georgia Department of Education, 156 Trinity Avenue, S.W., Atlanta 30303.

Aids to Media Selection for Students and Teachers, available from U.S. Department of Health, Education and Welfare, Office of Education, Bureau of Elementary and Secondary Education, Office of Libraries and Learning Resources, Washington, D.C.

Reviews and bibliographies of recommended mathematics resources and innovative program descriptions are published regularly in the journals and periodicals. The following titles are recommended.

Arithmetic Teacher, National Council of Teachers of Mathematics, monthly September-April.

Computing Teacher, The Oregon Council for Computer Education, six issues per year.

Creative Computing, Creative Computing Association, bimonthly.

Mathematics Teacher, National Council of Teachers of Mathematics, monthly September-May.

Science Books & Films, American Association for the Advancement of Science, quarterly.

School Science and Mathematics, School Science and Mathematics Association, monthly October-May.

The Georgia Department of Education provides resources and services which are available to teachers and students through their media center.

Educational Media Services Division, Georgia Department of Education, Items 1 and 2, 1066 Sylvan Road, S.W., Atlanta 30310; Item 3, 1540 Stewart Avenue, S.W., Atlanta 30310.

1. *Georgia Tapes for Teaching: Catalog of Classroom Teaching Tapes for Georgia Schools* (and supplements). Arranged by subject, this catalog lists the titles of *audio* tapes which on request will be duplicated. Recommended listening audiences are indicated. A one-time school registration is required. The requesting media center must provide the blank reel-to-reel or cassette tape; return postage is provided by the Georgia Department of Education.

2. *Catalog of Classroom Teaching Films for Georgia Schools* (and supplements). The 16mm films listed and annotated are arranged by titles but indexed by subjects; recommended viewing audiences are indicated. Registration (annual beginning in September or semiannual beginning in January) requires a minimal fee; each registration provides a specified weekly film quota, but multiple registrations are accepted. Many films are broadcast over the Georgia Educational Television Network and some may be duplicated on videotapes for later use. Information about this service and the broadcast schedule is provided annually to the system media contact person.

3. *Instructional Television Schedule*. Copies of the *Schedule* with series descriptions and broadcast times are available on request from your system media contact person, who also coordinates orders for needed teacher manuals. Although recommended viewing audiences are indicated, the *Schedule* and manuals should be examined for potential use of a program or series to introduce, develop or reinforce mathematics concepts.

Educational Information Center (ERIC), Georgia Department of Education, 212 State Office Building, Atlanta 30334.

Research service is provided to Georgia public school administrators and their central office staff. Computer and manual searches of Educational Resources Information Center (ERIC) data base which includes over 325,000 references to education documents related to exemplary projects and model teaching strategies can be requested by the media staff through the system media contact person.

Readers Services, Public Library Services, Georgia Department of Education, Atlanta 30334.

Books, pamphlets and periodicals about the teaching of mathematics and the various fields of mathematics are available for workshops and inservice activities as well as individual use.

"Selected List of Books for Teachers" (and supplements) and "Periodical List" (and supplements) identifying titles in the Public Library Services collection can be obtained by the school media specialist on request. Georgia Library Information Network (GLIN), another reference and bibliographic service, provides access to publications in the collections of approximately 150 participating public, special and academic libraries. Requests for these services and resources should be made through the local public libraries by the school media staff.

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CODES FOR REFERENCE LISTING

P	—Problem Solving
Ap	—Applications
S	—Strategies of Teaching
H	—History
S/N/N	—Sets, Numbers and Numeration
R/F	—Relations and Functions
O/P/N	—Operations, Their Properties and Number Theory
G	—Geometry
A	—Algebra
P/S	—Probability and Statistics
M/E	—Measurement and Estimation
C/C	—Computing and Computers
MR/L	—Mathematical Reasoning and Logic
C	—Calculus
O	—Other
	Personal Finance
	Analysis
	General Math

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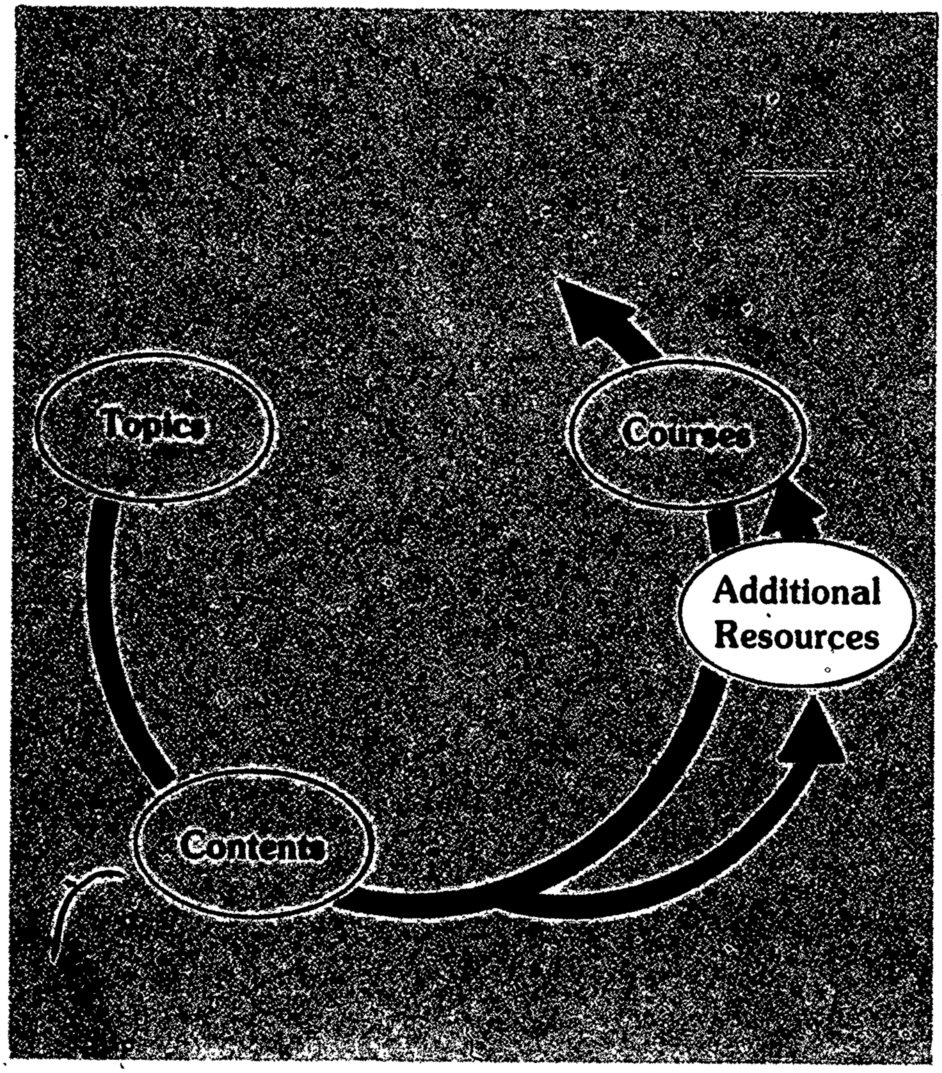

APPENDICES

Appendix A
Organizations for the Essentials of Education

Society must reaffirm the value of a balanced education.

Leaders of several professional organizations reached this conclusion in 1978. They circulated a statement on the essentials of education among a number of professional associations whose governing boards endorsed the statement and urged that it be called to the immediate attention of the entire education community, of policy makers and of the public at large.

The statement that follows embodies the collective concern of the endorsing associations. It expresses their call for a renewed commitment to a more complete and more fulfilling education for all.

The associations invite the concurrence, support and participation of everyone interested in education.

THE ESSENTIALS OF EDUCATION

Public concern about basic knowledge and the basic skills in education is valid. Society should continually seek out, define and then provide for every person those elements of education that are essential to a productive and meaningful life.

The basic elements of knowledge and skill are only a part of the essentials of education. In an era dominated by cries for going "back to the basics," for "minimal competencies," and for "survival skills," society should reject simplistic solutions and declare a commitment to the essentials of education.

A definition of the essentials of education should avoid three easy tendencies: to limit the essentials to "the three R's" in a society that is highly technological and complex; to define the essentials by what is tested at a time when tests are severely limited in what they can measure; and to reduce the essentials to a few "skills" when it is obvious that people use a combination of skills, knowledge and feelings to come to terms with their world. By rejecting these simplistic tendencies, educators will avoid concentration on training in a few skills at the expense of preparing students for the changing world in which they must live.

Educators should resist pressures to concentrate solely upon easy-to-teach, easy-to-test bits of knowledge, and must go beyond short-term objectives of training for jobs or producing citizens who can perform routine tasks but cannot apply their knowledge or skills, cannot reason about their society, and cannot make informed judgments.

What, then are the essentials of education?

Educators agree that the overarching goal of education is to develop informed, thinking citizens capable of participating in both domestic and world affairs. The development of such citizens depends not only upon education for citizenship, but also upon other essentials of education shared by all subjects.

The interdependence of skills and content is the central concept of the essentials of education. Skills and abilities do not grow in isolation from content. In all subjects, students develop skills in using language and other symbol systems; they develop the ability to reason; they undergo experiences that lead to emotional and social maturity. Students master these skills and abilities through observing, listening, reading, talking, and writing *about* science, mathematics, history and the social sciences, the arts and other aspects of our intellectual, social and cultural heritage. As they learn about their world and its heritage they necessarily deepen their skills in language and reasoning and acquire the basis for emotional, aesthetic and social growth. They also become aware of the world around them and develop an understanding and appreciation of the interdependence of the many facets of that world.

More specifically, the essentials of education include the ability to use language, to think, and to communicate effectively; to use mathematical knowledge and methods to solve problems; to reason logically; to use abstractions and symbols with power and ease; to apply and to understand scientific knowledge and methods; to make use of technology and to understand its limitations; to express oneself through the arts and to understand the artistic expressions of others; to understand other languages and cultures; to understand spatial relationships; to apply knowledge about health, nutrition, and physical activity; to acquire the capacity to meet unexpected challenges; to make informed value judgments; to recognize and to use one's full learning potential; and to prepare to go on learning for a lifetime.

Such a definition calls for a realization that all disciplines must join together and acknowledge their interdependence. Determining the essentials of education is a continuing process, far more demanding and significant than listing isolated skills assumed to be basic. Putting the essentials of education into practice requires instructional programs based on this new sense of interdependence.

Educators must also join with many segments of society to specify the essentials of education more fully. Among these segments are legislators, school boards, parents, students, workers' organizations, businesses, publishers, and other groups and individuals with an interest in education. All must now participate in a coordinated effort on behalf of society to confront this task. *Everyone* has a stake in the essentials of education.

Professional Associations Endorsing This Statement

American Alliance for Health, Physical Education, Recreation and Dance
1201 16th Street, N.W., Washington, D.C. 20036 (202) 833-5553

American Council on the Teaching of Foreign Languages
2 Park Avenue, Room 1814, New York, New York 10016 (212) 689-8021

Association for Supervision and Curriculum Development
225 North Washington Street, Alexandria, Virginia 22314 (703) 549-9110

International Reading Association
800 Barksdale Road, P.O. Box 8139, Newark, Delaware 19711 (302) 731-1600

Music Educators National Conference
1902 Association Drive, Reston, Virginia 22091 (703) 860-4000

National Art Education Association
1916 Association Drive, Reston, Virginia 22091 (703) 860-8000

National Association of Elementary School Principals
1801 N. Moore Street, Arlington, Virginia 22209 (703) 528-6000

National Council for the Social Studies
3615 Wisconsin Avenue, N.W., Washington, D.C. 20016 (202) 966-7840

National Council of Teachers of English
1111 Kenyon Road, Urbana, Illinois 61801 (217) 328-3870

National Council of Teachers of Mathematics
1906 Association Drive, Reston, Virginia 22091 (703) 620-9840

National Science Teachers Association
1742 Connecticut Avenue, N.W., Washington, D.C. 20009 (202) 265-4150

Speech Communication Association
5205 Leesburg Pike, Falls Church, Virginia 22041 (703) 379-1888

Appendix B
Position Paper on Basic Mathematical Skills of the National Council of Supervisors of Mathematics

INTRODUCTION

The currently popular slogan "Back to the Basics" has become a rallying cry of many who perceive a need for certain changes in education. The result is a trend that has gained considerable momentum and has initiated demands for programs and evaluations which emphasize narrowly defined skills.

Mathematics educators find themselves under considerable pressure from boards of education, legislatures, and citizens' groups who are demanding instructional programs which will guarantee acquisition of computational skills. Leaders in mathematics education have expressed a need for clarifying what are the basic skills needed by students who hope to participate successfully in adult society.

The narrow definition of basic skills which equates mathematical competence with computational ability has evolved as a result of several forces:

1. Declining scores on standardized achievement tests and college entrance examinations;

2. Reactions to the results of the National Assessment of Educational Progress;

3. Rising costs of education and increasing demands for accountability;

4. Shifting emphasis in mathematics education from curriculum content to instructional methods and alternatives.

5. Increased awareness of the need to provide remedial and compensatory programs;

6. The widespread publicity given to each of the above by the media.

This widespread publicity, in particular, has generated a call for action from governmental agencies, educational organizations and community groups. In responding to these calls, the National Institute of Education adopted the area of basic skills as a major priority. This resulted in a Conference on Basic Mathematical Skills and Learning, held in Euclid, Ohio, in October, 1975.

The National Council of Supervisors of Mathematics (NCSM), during the 1976 Annual Meeting in Atlanta, Georgia, met in a special session to discuss the Euclid Conference Report. More than 100 members participating in that session expressed the need for a unified position on basic mathematical skills which would enable them to provide more effective leadership within their respective school systems, to give adequate rationale and direction in their tasks of implementing basic mathematics programs, and to appropriately expand the definition of basic skills. Hence, by an overwhelming majority, they mandated the NCSM to establish a task force to formulate a position on basic mathematical skills. This statement is the result of that effort.

RATIONALE FOR THE EXPANDED DEFINITION

There are many reasons why basic skills must include more than computation. The present technological society requires daily use of such skills as estimating, problem solving, interpreting data, organizing data, measuring, predicting and applying mathematics to everyday situations. The changing needs of society, the explosion of the amount of quantitative data, and the availability of computers and calculators demand a redefining of the priorities for basic mathematics skills. In recognition of the inadequacy of computation alone, NCSM is going on record as providing both a general list of basic mathematical skills and a clarification of the need for such an expanded definition of basic skills.

Any list of basic skills must include computation. However, the role of computational skills in mathematics must be seen in the light of the contributions they make to one's ability to use mathematics in everyday living. In isolation, computational skills contribute little to one's ability to participate in mainstream society. Combined effectively with the other skill areas, they provide the learner with the basic mathematical ability needed by adults.

DEFINING BASIC SKILLS

The NCSM views basic mathematical skills as falling under ten vital areas. The ten skill areas are interrelated and many overlap with each other and with other disciplines. All are basic to pupils' development of the ability to reason effectively in varied situations.

This expanded list is presented with the conviction that mathematics education must not emphasize computational skills to the neglect of other critical areas of mathematics. The ten components of basic mathematical skills are listed below, but the order of their listing should not be interpreted as indicating either a priority of importance or a sequence for teaching and learning.

Furthermore, as society changes our ideas about which skills are basic also change. For example, today our students should learn to measure in both the customary and metric systems, but in the future the significance of the customary system will be mostly historical. There will also be increasing emphasis on when and how to use hand-held calculators and other electronic devices in mathematics.

TEN BASIC SKILL AREAS

Problem Solving

Learning to solve problems is the principal reason for studying mathematics. Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in texts is one form of problem solving, but students also should be faced with non-textbook problems. Problem-solving strategies involve posing questions, analyzing situations, translating results, illustrating results, drawing diagrams, and using trial and error. In solving problems, students need to be able to apply the rules of logic necessary to arrive at valid conclusions. They must be able to determine which facts are relevant. They should be unfearful of arriving at tentative conclusions and they must be willing to subject these conclusions to scrutiny.

Applying Mathematics to Everyday Situations

The use of mathematics is interrelated with all computation activities. Students should be encouraged to take everyday situations, translate them into mathematical expressions, solve the mathematics, and interpret the results in light of the initial situation.

Alertness to the Reasonableness of Results

Due to arithmetic errors or other mistakes, results of mathematical work are sometimes wrong. Students should learn to inspect all results and to check for reasonableness in terms of the original problem. With the increase in the use of calculating devices in society, this skill is essential.

Estimation and Approximation

Students should be able to carry out rapid approximate calculations by first rounding off numbers. They should acquire some simple techniques for estimating quantity, length, distance, weight, etc. It is also necessary to decide when a particular result is precise enough for the purpose at hand.

Appropriate Computational Skills

Students should gain facility with addition, subtraction, multiplication, and division with whole numbers and decimals. Today it must be recognized that long, complicated computations will usually be done with a calculator. Knowledge of single-digit number facts is essential and mental arithmetic is a valuable skill. Moreover, there are everyday situations which demand recognition of, and simple computation with, common fractions.

Because consumers continually deal with many situations that involve percentage, the ability to recognize and use percents should be developed and maintained.

Geometry

Students should learn the geometric concepts they will need to function effectively in the 3-dimensional world. They should have knowledge of concepts such as point, line, plane, parallel, and perpendicular. They should know basic properties of simple geometric figures, particularly those properties which relate to measurement and problem-solving skills. They also must be able to recognize similarities and differences among objects.

Measurement

As a minimum skill, students should be able to measure distance, weight, time, capacity, and temperature. Measurement of angles and calculations of simple areas and volumes are also essential. Students should be able to perform measurement in both metric and customary systems using the appropriate tools.

Reading, Interpreting, and Constructing Tables, Charts, and Graphs

Students should know how to read and draw conclusions from simple tables, maps, charts and graphs. They should be able to condense numerical information into more manageable or meaningful terms by setting up simple tables, charts, and graphs.

Using Mathematics to Predict

Students should learn how elementary notions of probability are used to determine the likelihood of future events. They should learn to identify situations where immediate past experience does not affect the likelihood of future events. They should become familiar with how mathematics is used to help make predictions such as election forecasts.

Computer Literacy

It is important for all citizens to understand what computers can and cannot do. Students should be aware of the many uses of computers in society, such as their use in teaching/learning, financial transactions, and information storage and retrieval. The "mystique" surrounding computers is disturbing and can put persons with no understanding of computers at a disadvantage. The increasing use of computers by government, industry, and business demands an awareness of computer uses and limitations.

BASIC SKILLS AND THE STUDENT'S FUTURE

Anyone adopting a definition of basic skills should consider the "door-opening/door-closing" implications of the list. The following diagram illustrates expected outcomes associated with various amounts of skill development.

Scope of Skill Development		Expected Outcomes
EXPANDED SKILLS Mathematical skills beyond those described here plus a desire to learn more.	➡	**POTENTIAL LEADERS** Employment and educational opportunities will continue to increase as mathematical skills continue to grow.
BASIC SKILLS The skills described here.	➡	**EMPLOYMENT VERY LIKELY** Employment opportunities are predictable. Doors to further education opportunities are open.
MINIMAL SKILLS Limited skills, primarily computation. Little exposure to the other skill areas described here.	➡	**LIMITED OPPORTUNITIES** Unemployment likely. Potential generally limited to low-level jobs.

MINIMUM ESSENTIALS FOR HIGH-SCHOOL GRADUATION

Today some school boards and state legislatures are starting to mandate mastery of minimum essential skills in reading and mathematics as a requirement for high-school graduation. In the process, they should consider the potential pitfalls of doing this without an appropriate definition of "basic skills." If the mathematics requirements are set inordinately high, then a significant number of students may not be able to graduate. On the other hand, if the mathematics requirements are set too low and mathematial skills are too narrowly defined, the result could be a sterile mathematics program concentrating exclusively on learning of low-level mathematical skills. This position paper neither recommends nor condemns minimal competencies for high-school graduation. However, the ten components of basic skills stated here can serve as guidelines for state and local school systems that are considering the establishment of minimum essential graduation requirements.

DEVELOPING THE BASIC SKILLS

One individual difference among students is style or way of learning. In offering opportunities to learn the basic skills, options must be provided to meet these varying learning styles. The present "back-to-basics" movement may lead to an emphasis on drill and practice as a way to learn.

Certainly drill and practice is a viable option, but it is only one of many possible ways to bring about learning and to create interest and motivation in students. Learning centers, contracts, tutorial sessions, individual and small-group projects, games, simulations and community-based activities are some of the other options that can provide the opportunity to learn basic skills. Furthermore, to help students fully understand basic mathematical concepts, teachers should utilize the full range of activities and materials available, including objects the students can actually handle.

The learning of basic mathematical skills is a continuing process which extends through all of the years a student is in school. In particular, a tendency to emphasize computation while neglecting the other nine skill areas at the elementary level must be avoided.

EVALUATING AND REPORTING STUDENT PROGRESS

Any systematic attempt to develop basic skills must necessarily be concerned with evaluating and reporting pupil progress.

In evaluation, test results are used to judge the effectiveness of the instructional process and to make needed adjustments in the curriculum and instruction for the individual student. In general, both educators and the public have accepted and emphasized an overuse of and overconfidence in the results of standardized tests. Standardized tests yield comparisons between students and can provide a rank ordering of individuals, schools, or districts. However, standardized tests have several limitations including the following:

a. Items are not necessarily generated to measure a specific objective or instructional aim.

b. The tests measure only a sample of the content that makes up a program; certain outcomes are not measured at all.

Because they do not supply sufficient information about how much mathematics a student knows, standardized tests are not the best instruments available for reporting individual pupil growth. Other alternatives such as criterion tests or competency tests must be considered. In criterion tests, items are generated which measure the specific objectives of the program and which establish the student's level of mastery of these objectives. Competency tests are designed to determine if the individual has mastered the skills necessary for a certain purpose such as entry into the job market. There is also need for open-ended assessments such as observations, interviews, and manipulative tasks to assess skills which paper and pencil tests do not measure adequately.

Reports of pupil progress will surely be made. But, while standardized tests will probably continue to dominate the testing scene for several years, there is an urgent need to begin reporting pupil progress in other terms, such as criterion tests and competency measures. This will also demand an immediate and extensive program of inservice education to instruct the general public on the meaning and interpretation of such data and to enable teachers to use testing as a vital part of the instructional process.

Large scale testing, whether involving all students or a random sample, can result in interpretations which have great influence on curriculum revisions and development. Test results can indicate, for example, that a particular mathematical topic is being taught at the wrong time in the student's development and that it might better be introduced later or earlier in the curriculum. Or, the results might indicate that students are confused about some topic as a result of inappropriate teaching procedures. In any case, test results should be carefully examined by educators with special skills in the area of curriculum development.

CONCLUSION

The present paper represents a preliminary attempt by the National Council of Supervisors of Mathematics to clarify and communicate its position on basic mathematical skills. The NCSM position establishes a framework within which decisions on program planning and implementation can be made. It also sets forth the underlying rationale for identifying and developing basic skills and for evaluating pupils' acquisition of these competencies. The NCSM position underscores the fundamental belief of the National Council of Supervisors of Mathematics that any effective program of basic mathematical skills must be directed not "back" but *forward* to the essential needs of adults in the present and future.

You are encouraged to make and distribute copies of this paper.

MEMBERS OF NCSM TASK FORCE ON BASIC MATHEMATICAL SKILLS

Mildred Daniels, Detroit Public Schools, Detroit, MI
David Dye, Minnesota Department of Education, St. Paul, MN
Jane Donnelly Gawronski, San Diego County Department of Education, San Diego, CA
William J. Geppert, Delaware Department of Public Instruction, Dover, DE
President, Association of State Supervisors of Mathematics
Peggy A. House, University of Minnesota, Minneapolis, MN
Elgin Schilhab, Austin Independent School District, Austin, TX
Glyn Sharpe, Jefferson County Public Schools, Lakewood, CO
Albert P. Schulte, Oakland Public Schools, Pontiac, MI
Dorothy Strong, Chicago Board of Education, Chicago, IL
B. Ross Taylor, Minneapolis Public Schools, Minneapolis, MN
NCSM Basic Skills Task Force Chairperson
Alex Tobin, School District of Philadelphia, Philadelphia, PA
President, NCSM
Kay Tobin, Lexington Public Schools, Lexington, MA
Edward Esty, National Institute of Education, Washington, D.C.
Resource Person, National Institute of Education

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Appendix C
NCTM-MAA Position Statement
Recommendations for the Preparation of High School Students for College Mathematics Courses

The following statement, adopted by the Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics, is a brief outline of the basic ingredients of adequate preparation for collegiate-level mathematics.* The statement does not break new ground; it reflects standards that have been generally accepted for over a decade. It is intended to support the continuing efforts of conscientious teachers everywhere to provide students with sound and stimulating mathematical training. It is specifically designed to provide a benchmark for our efforts and those of others to assess and react to recent reports of a general decline in the performance of students in mathematics.

A joint committee of the Mathematical Association of America and the National Council of Teachers of Mathematics consulted with secondary school and college teachers in various parts of the country to study recent trends in the preparation of students. The comments from these consultations on which there was strongest consensus are the basis for this statement and its ten recommendations.

The Mathematical Association of America and the National Council of Teachers of Mathematics wish to emphasize that the statement and recommendations, as they refer to secondary school programs, are addressed only to those programs for students planning to go to college and that they are not intended to be more comprehensive. During the past twenty years many important changes have taken place in both the content and teaching of mathematics at the secondary school level. Many excellent new programs have been adopted and taught effectively by teachers in elementary and secondary schools. Nevertheless, any consideration of the relative merits of new versus traditional school curricula has been deliberately avoided. A study of this issue would have exceeded both the charge to the committee and its limited resources. This statement and these recommendations incorporate many of the best features of both of these curricula and are addressed to all mathematics programs regardless of pedagogical heritage.

Necessary Course Work

Mathematics is a highly structured subject in which various concepts and techniques are greatly dependent on each other. The concepts of arithmetic and algebra, however, are basic to all of mathematics. Further work in mathematics and in all areas in which mathematics is used as a tool requires correct performance, with understanding, of basic arithmetic operations, the manipulation of algebraic symbols, and an understanding of what the manipulations mean.

Any student who is unable to perform arithmetic calculations and algebraic operations with accuracy and reasonable speed, to understand which operations to use in a given problem, and to determine whether the results have meaning is severely handicapped in the study and applications of mathematics. The prevalence of inexpensive pocket calculators makes the performance of complicated calculations less tedious, but the use of calculators does not lessen the need for students

*Collegiate mathematics refers to courses in calculus (or calculus and analytic geometry), probability and statistics, finite mathematics, and higher-level mathematics courses.

to understand which concepts and operations are needed to solve a problem, to make sensible estimates, and to analyze their results.

For further work in mathematics, and in many other areas from business to psychology, from biology to engineering, the ability to use algebra with skill and understanding is also essential. Having a passing grade in algebra is not enough. Both understanding and competence in the skills of algebra are necesary. Neither conceptual understanding nor technical skill alone will suffice in today's world, let alone in tomorrow's. Algebra is a useful subject which will help to solve problems in the real world. Opportunities to apply algebraic skills should be provided whenever possible, especially to problems that show the utility of mathematics.

Algebra courses in secondary school should include, in addition to the basic topics —

(a) polynomial functions;
(b) properties of logarithms;
(c) exponential and logarithmic functions and equations;
(d) arithmetic and geometric sequences and series;
(e) the binomial theorem;
(f) infinite geometric series;
(g) linear and quadratic inequalities.

For most students, adequate coverage of the topics in algebra requires at least two years of study.

Students who will take calculus — and this now includes many students who will take college work in business, premedicine, economics, biology, statistics, engineering, and physical science — may or may not need trigonometry, depending on the type of calculus course appropriate for their particular programs. But they will need a good deal of what is often called precalculus, including especially a sound understanding of the concept of a function, which is also fundamental for work beyond the most elementary level in probability and computing.

Those students needing trigonometry should study —

(a) trigonometric functions and their graphs;
(b) degree and radian measure;
(c) trigonometric identities and equations;
(d) inverse trigonometric functions and their graphs.

For such students, the equivalent of one semester should be devoted to the study of the topics in trigonometry.

All students who go on to take collegiate mathematics will find their college work easier if they have been introduced to some axiomatic system and to deductive reasoning. Traditionally this has been accomplished in a geometry course. Geometry courses in secondary school should include, in addition to basic topics —

(a) fundamental properties of geometric figures in three dimensions;
(b) applications of formulas for areas and volumes;
(c) experience in visualizing three-dimensional figures.

Other courses (the word *course* refers here and elsewhere in this statement to a semester course unless otherwise noted) beyond algebra, trigonometry, and geometry should be available to students who have adequate background and time to take them. A course in coordinate (or analytic) geometry is ideal, since it combines algebra with geometry and provides a useful preparation for calculus. In addition to coordinate geometry, courses in the following topics are valuable:

probability, statistics, elementary finite mathematics (or linear algebra), an introduction to computers and computing, and applications of mathematics.

If coordinate geometry is offered, it should include, in addition to the basic topics —

(a) conic sections;
(b) rational functions and their graphs;
(c) polar coordinates;
(d) parametric equations and their graphs.

Inductive as well as deductive reasoning, techniques of estimation and approximation, and an awareness of problem-solving techniques, with special emphasis on the transition from the verbal form to the language of mathematics, should be emphasized in all courses.

Calculus, where offered in secondary schools, should be at least a *full year* course and be taken only by those students who are strongly prepared in algebra, geometry, trigonometry, and coordinate geometry.

We recognize that many secondary schools have a curriculum similar to that outlined above. We emphasize again that, in order to be properly prepared for collegiate-level courses in mathematics, students need to develop skills (1) in applying standard techniques and (2) in understanding important concepts.

Recommendations

The Board of Governors of the Mathematical Association of America and the Board of Directors of the National Council of Teachers of Mathematics make the following recommendations:

1. Proficiency in mathematics cannot be acquired without individual practice. We therefore endorse the common practice of making regular assignments to be completed outside of class. We recommend that parents encourage their children to set aside sufficient time each day to complete these assignments and that parents actively support the request of the teacher that homework be turned in. Students should be encouraged to develop good study habits in mathematics courses at all levels and should develop the ability to read mathematics.

2. Homework and drill are very important pedagogical tools used to help the student gain understanding as well as proficiency in the skills of arithmetic and algebra, but students should not be burdened with excessive or meaningless drill. We therefore recommend that teachers and authors of textbooks step up their search for interesting problems that provide the opportunity to apply these skills. We realize that this is a difficult task, but we believe that providing problems that reinforce manipulative skills should have high priority, especially those that show that mathematics helps solve problems in the real world.

3. We are aware that teachers must struggle to maintain standards of performance in courses at all levels from kindergarten through college and that serious grade inflation has been observed. An apparently growing trend to reward effort or attendance rather than achievement has been making it increasingly difficult for mathematics teachers to maintain standards. We recommend that mathematics departments review evaluation procedures to ensure that grades reflect student achievement. Further, we urge administrators to support teachers in this endeavor.

4. In light of recommendation 3, we also recognize that the advancement of students without appropriate achievement has a detrimental effect on the individual student and on the entire

class. We therefore recommend that school districts make special provisions to assist students when deficiencies are *first* noted.

5. We recommend that cumulative evaluations be given throughout each course, as well as at its completion, to *all* students. We believe that the absence of cumulative evaluation promotes short-term learning. We strongly oppose the practice of exempting students from evaluations.

6. We recommend that computers and minicalculators be used in imaginative ways to reinforce learning and to motivate the student as proficiency in mathematics is gained. Calculators should be used to supplement rather than to supplant the study of necessary computational skills.

7. We recommend that colleges and universities administer placement examinations in mathematics prior to final registration to aid students in selecting appropriate college courses.

8. We encourage the continuation or initiation of joint meetings of college and secondary school mathematics instructors and counselors in order to improve communication concerning mathematics prerequisites for careers, the preparation of students for collegiate mathematics courses, joint curriculum coordination, remedial programs in schools and colleges, the exchange of successful instructional strategies, the planning of in-service programs, and other related topics.

9. Schools should frequently review their mathematics curriculum to see that it meets the needs of its students in preparing them for college mathematics. School districts that have not conducted a curriculum analysis recently should do so now, primarily to identify topics in the curriculum that could be either omitted or de-emphasized, if necessary, in order to provide sufficient time for the topics included in this statement. We suggest, for example, that the following could be de-emphasized or omitted from the curriculum:

   (a) Logarithmic calculations that can better be handled by calculators or computers

   (b) The extensive solving of triangles in trigonometry

   (c) Proofs of superfluous or trivial theorems in geometry

10. We recommend that algebraic concepts and skills be incorporated wherever possible into geometry and other courses beyond algebra to help students retain these concepts and skills.

This position statement was prepared jointly by the National Council of Teachers of Mathematics, 1906 Association Dr., Reston, VA 22091, and the Mathematical Association of America, 1225 Connecticut Ave., NW, Washington, DC 20036.

Appendix D
Correlation of Georgia Statewide Basic Skills Test Indicator Clusters and Secondary School Mathematics Collection C Objectives

The Georgia Statewide Basic Skills Tests for Mathematics and Problem Solving are based upon indicator clusters or descriptive statements. The indicator clusters listed here include assessment characteristics which the item writers and content review committees used while developing the item pool for the first edition of these tests. The items were designed to measure minimal skills and skill application in the context of every day and real world situations. Since there will be several test administrations before applying standards and high school graduation based on test performance, it is possible that further refinement or clarification of indicator clusters may occur as the assessment instruments are used and evaluated.

Special attention should be given to those indicator clusters listed for the area of problem solving since they include mathematics skills. While a student's competency in problem solving will be evaluated and described separately, the measurement approach in the statewide tests frequently will be in the context of both reading and mathematics problems since the skills are so interrelated.

Each indicator cluster is keyed to one or more objectives of Collection C and the strands of the curriculum guide, *Mathematics for Georgia Secondary Schools*. Collection C objectives include the knowledge, skills and processes presumed necessary for productive citizenship. The strands or topics in secondary school mathematics are coded as follows.

S/N/N —Sets, Numbers and Numeration

R/F —Relations and Functions

O/P/N —Operations, Their Properties and Number Theory

G —Geometry

A —Algebra

P/S —Probability and Statistics

M/E —Measurement and Estimation

C/C —Computing and Computers

MR/L —Mathematical Reasoning and Logic

MATHEMATICS

Indicator Clusters and Assessment Characteristics	**Collection C Objective Numbers and Strand Codes**
• **NUMBER CONCEPTS**	
Indicator Cluster 1	4(S/N/N, C/C, M/E)
The student translates from words to numerals and the reverse.	
Assessment Characteristics	
In the assessment of this cluster whole numbers and decimals are appropriate; however, fractions or percents should not be included for conversion.	

Indicator Cluster 2 3(S/N/N, R/F)

The student orders fractions, decimals or percents.

Assessment Characteristics

Items used for assessment are mutually exclusive, not involving combinations of these number concepts. If fractions are included they are limited to halves through tenths, plus twelfths and hundredths, not sixths, sevenths or ninths. Mixed numbers can be used; however, improper fractions are not appropriate.

Indicator Cluster 3 4(S/N/N, C/C, M/E)

The student translates from decimals to percents and the reverse.

Assessment Characteristics

The student's understanding of the conversion process of these number concepts in any given context is the primary consideration. For this cluster, the use of decimals over one, smaller than hundredths; percents with fractions or decimals and percents over 100 are not suitable.

Indicator Cluster 4 4(S/N/N, C/C, M/E)

The student translates from fractions to percents and the reverse.

Assessment Characteristics

In this cluster, assessment includes percents with fractions in combination and repeating decimals. Fractions are limited to halves, thirds, fourths, fifths, eighths, tenths and hundredths. The use of mixed numbers and percents over 100 are not suitable.

Indicator Cluster 5 4(S/N/N, C/C, M/E)

The student translates from fractions to decimals and the reverse.

Assessment Characteristics

In measuring the student's ability to achieve this cluster, improper fractions and decimals smaller than thousandths are not appropriate. Suitable means include the use of mixed numbers and repeating decimals.

NUMBER OPERATIONS

Indicator Cluster 6 5(O/P/N, C/C)

The student selects appropriate operations for a given problem situation.

Assessment Characteristics

Money and common decimals are among the productive subject areas for the measurement of this cluster.

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Indicator Cluster 7 — 6(C/C), 7(S/N/N, O/P/N, C/C, R/F)

The student computes with whole numbers, fractions, decimals and percents.

Assessment Characteristics

The assessment of this cluster may include horizontal and vertical presentations using mixed numbers, like and unlike denominations, simplifying fractions. Identity and inverse properties, improper fractions, percents over 100 or less than one, percents of increase or decrease are to be excluded. The use of graphics and word problems in items are not suitable for this indicator.

Indicator Cluster 8 — 8(O/P/N, C/C), 9(O/P/N, C/C)

The student applies properties of operations.

Assessment Characteristics

This cluster assesses a student's application of the identity, inverse, commutative, associative and distributive properties of operations. The identification of a particular property is not the objective in this case; therefore, the selection of the correct option should not be predicated on same.

Indicator Cluster 9 — 34(All), 37(All), 38(All)

The student solves simple word problems.

Assessment Characteristics

Computation, purposely, is to be kept simple. The assessment of this cluster includes asking for the solution equation only (not the answer) and problems demanding computation. No academic word problems or problems involving percent of increase or decrease are to be used. Sales tax and changing recipes are among suggested contexts for this cluster.

- **RELATIONS AND FORMULAS**

Indicator Cluster 10 — 19(R/F, G, M/E, C/C), 22(G, M/E)

The student applies proportions.

Assessment Characteristics

The aim of this cluster is to determine the student's ability in the application of proportional relationships. Assessment includes the use of similar drawings or scale drawings as well as unit pricing and "better buys" concept in item presentation.

Indicator Cluster 11 — 23(G, M/E, R/F, C/C)

The student applies formulas.

Assessment Characteristics

Formulas such as simple interest, area-circumference, distance/rate, miles per gallon and perimeter are appropriate in the assessment of a student's ability in formula application. Complex formulas which would include the Pythagorean theorem and compound interest *are not to be in the measurement* of this indicator. Formulas may be supplied.

- **STATISTICS**

Indicator Cluster 12 — 36(P/S, C/C)

The student computes the mean and median.

Assessment Characteristics

For any set of numbers, in the assessment of this cluster, the mean and the median should be different. The median is determined from a set with an odd number of elements which may be arranged in order. The mean should be a whole number.

Indicator Cluster 13 — 32(S/N/N, P/S)

The student determines probabilities.

Assessment Characteristics

Events with zero or one probability are suitable assessment areas for this cluster. Limitations include the presentation of probabilities as a percent; joint (and/or), independent and dependent events; and the use of combinations of permutations.

Indicator Cluster 14 — 28(S/N/N, R/F, P/S)

The student organizes data into tables, charts and graphs.

Assessment Characteristics

Assessment problems in this cluster involve the selection of the appropriate representation of the data as well as some interpretation. Graphs, circle graphs and pictographs, which may be incorrectly labeled or have missing information.

Indicator Cluster 15 — 14(R/F)
29(R/F, P/S, C/C, M/E)
31(P/S, C/C)

The student interprets data in the form of tables, charts and graphs.

Assessment Characteristics

The ability of a student to discover a relationship or rule from the presented material is assessed in this cluster. Formats wherein data can be interpreted may include circle graphs, bar graphs, line graphs and pictographs.

- **MEASUREMENT AND ESTIMATION**

Indicator Cluster 16 — 24(R/F, M/E)
27(M/E)

The student identifies customary or metric units to measure length, area, volume, weight, time and temperature.

Assessment Characteristics

This cluster involves choosing the unit that applies to a specific type of measurement such as picking the appropriate type or size unit. It is not suitable to use conversions from metric to customary or the reverse in this indicator. Units appropriate include grams, meters, liters, Celsius, inches, feet, yards, miles, ounces, pounds, pints, quarts, gallons, Fahrenheit, seconds, minutes, hours, days, weeks or months. Prefixes such as milli-, centi- and kilo- are suitable for inclusion.

Indicator Cluster 17 — 24(M/E)

The student applies customary or metric units of measurement to determine length, area, volume, weight, time and temperature.

Assessment Characteristics

Nonstandard units of measurement can be used in the assessment of this cluster. It specifically involves presenting a measurement scale and having the student identify or apply it. Conversion from metric to customary or the reverse is not appropriate; however, conversion within a system of measurement can be included. Units appropriate include grams, meters, liters, Celsius, inches, feet, yards, miles, ounces, pounds, pints, quarts, gallons, Fahrenheit, seconds, minutes, hours, days, weeks or months. Prefixes such as milli-, centi- and kilo- are suitable for inclusion.

Indicator Cluster 18 — 11(S/N/N, R/F, O/P/N, M/E, C/C) 12(O/P/N, A, G, P/S, M/E, C/C, MR/L) 26(M/E)

The student estimates numbers (results) using round numbers, with or without units of measurement.

Assessment Characteristics

The objective of this cluster is specifically to assess the student's ability to estimate a result. A nonstandard unit of measurement can be used to ask the student to estimate the number of units contained in a drawing in practical settings.

Indicator Cluster 19 — 4(S/N/N, C/C, M/E)

The student determines amounts of money.

Assessment Characteristics

The assessment of this cluster can include making change by counting or by subtracting, as well as determining the least number of coins. Computation can be involved, as well as simply showing an amount of money. Exclude the use of half dollars, silver dollars, or two-dollar bills.

- **GEOMETRY**

Indicator Cluster 20 — 15(G)

The student identifies sets of points using standard names.

Assessment Characteristics

Sets of points in the assessment of this cluster may include identification of the circle, triangle, rectangle, point, line, plane, parallelogram, cone, sphere, cylinder, pyramid and cube.

Indicator Cluster 21	16(G) 17(G) 18(G)

The student identifies geometric relations and properties.

Assessment Characteristics

Geometric relations and properties to be identified in this cluster may include: parallel, perpendicular, similar, congruent, vertical and horizontal. The concepts of congruent and similar are to be measured and are not to be presented as vocabulary items. Additionally, degrees in a right angle, triangle, circle and rectangle are included.

Indicator Cluster 22	20(R/F, G)

The student identifies points of Cartesian coordinates.

Assessment Characteristics

The assessment of this cluster includes finding the coordinates of a point. An appropriate strategy for cluster measurement may be the use of a street map.

PROBLEM SOLVING

Indicator Clusters and Assessment Characteristics	Collection C Objective Numbers and Strand Codes

- **COMPONENT SKILLS**

Indicator Cluster 1	(39(MR/L)

The student distinguishes between fact and opinion.

Assessment Characteristics

This cluster is assessed with such materials as editorials, books, movies and news reports. Stems must present a problem context. Minimal prior information (i.e., information other than that presented in the stem or associated stimulus material) should be required for correct responding. Statements of values are not considered facts.

Indicator Cluster 2	30(S/N/N, R/F, P/S, C/C) 4(S/N/N, R/F, MR/L)

The student recognizes main ideas, details, sequences of events and cause and effect relationships.

Assessment Characteristics

Includes explicit or implicit statement of ideas, details, sequences and relationships. Correct responding may require

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prior knowledge; however, answer must be dependent on item stimulus material. The details of events selected for item content must be necessary or relevant to overall comprehension of the problem situation. Items requiring identification of appropriate statement of a problem are included in this cluster.

Indicator Cluster 3 — 37(All)

The student recognizes appropriate reference sources.

Assessment Characteristics

This cluster assesses the student's ability to identify various reference sources such as a card catalog, an encyclopedia, types of directories and general library skills. Also included are yellow pages, classified ads, recipes/cookbooks and instructional manuals. Some items should focus on *why* one source is more appropriate than another for a specific task. Items preferably should emphasize the use of non-academic reference sources.

Indicator Cluster 4 — 37(All)

The student locates information in reference materials.

Assessment Characteristics

This cluster assesses the student's ability to use various sources of information, including library reference materials. Items requiring use of cross-references, multiple-step search strategies, and recognition of various classification schemes are included in this cluster.

Indicator Cluster 5 — 11(S/N/N, R/F, O/P/N, M/E, C/C) 12(O/P/N, A, G, P/S, M/E, C/C, MR/L) 26(M/E)

The student estimates outcomes, with or without units of measurement.

Assessment Characteristics

The objective of this cluster is specifically to assess a student's ability to estimate a result. Appropriate items include use of estimation in planning stages of problem solving as well as items asking which of several problem-solving methods gives the best estimate of quantity or other result. Items requiring identification of alternative solution strategies and items requiring value judgments about the appropriateness of alternative solution strategies are included.

Indicator Cluster 6 — 39(MR/L)

The student draws conclusions.

Assessment Characteristics

Conclusions are considered the result of a deductive or inductive reasoning process. The conclusion may act as a summary statement, account for a synthesis of the information or bring closure to the passage. In most cases, multiple pieces of information on which to base a conclusion are included in the passage. Items requiring identification of valid conclusions are appropriate for this cluster. Where conclusions involve predictions, generalizations or comparisons, these must be stated in the item stem or associated stimulus material. Also appropriate are items requiring the student to identify or state a problem or question to be resolved. This indicator may be assessed in the context of free-response items.

- **DATA FLUENCY**

Indicator Cluster 7 — 38(All)

The student interprets nongraphic instructions, labels, forms and applications.

Assessment Characteristics

The emphasis of this cluster is application rather than terminology. Items represented are actual forms or other information presented in practical situations. Item content may include transportation, occupational and career information.

Indicator Cluster 8 — 37(All), 39(MR/L)

The student recognizes relevance of data.

Assessment Characteristics

Items pertaining to this cluster must require the student to identify relevant or irrelevant pieces of information for a specific problem situation and set of resolution criteria. Particularly, the student will identify what further piece(s) of information may be necessary to respond to a task or question or identify unnecessary information which may cause confusion or be extraneous to the situation.

Indicator Cluster 9 — 28(S/N/N, R/F, P/S)

The student organizes data into tables, charts and graphs.

Assessment Characteristics

Assessment problems in this cluster involve the selection of the appropriate representation of data for a specific purpose or set of resolution criteria. Items may also require construction of decision tables or flow charts. Content of items may include labeling, transportation, career and occupational information. Items should focus on the organization of data in order to facilitate problem solution. This indicator may be assessed in the context of free-response items.

Indicator Cluster 10 — 29(R/F, P/S, C/C, M/E), 30(S/N/N, R/F, P/S, C/C)

The student interprets data in the form of tables, charts and graphs.

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Assessment Characteristics

Items should require a student to identify a relationship or rule from the presented material. Formats wherein data can be interpreted may include circle graphs, bar graphs, line graphs and pictographs. Items requiring the use of decision tables and flow charts are appropriate. Items may include labeling, transportation, career and occupational information.

• **MODEL APPLICATION**

Indicator Cluster 11

33(P/S)
40(S/S/N, R/F, MR/L)

The student makes predictions, generalizations and comparisons.

Assessment Characteristics

Prediction implies a future event; a degree of probability exists. Generalizations are the result of inductive reasoning; specifics or details are presented from which the general statement is derived. Deduction may be involved as well. Comparisons are made based on some defined variable which is constant for that comparison and should be required with respect to some specific criteria. Items may entail explicit or implicit problem resolution criteria. This indicator may be assessed in the context of free-response items.

Indicator Cluster 12

34(All)
37(All)
38(All)

The student solves simple word problems.

Assessment Characteristics

Problem solutions may involve several operations performed in a specified or implied sequence. Solutions may require responses for which there are no readily apparent response cues in the item stem or associated stimulus material (e.g., common knowledge responses). Some items may require value judgments about the appropriateness of alternative solution strategies. Solutions are not necessarily numerical results.

CAREERS IN MATHEMATICS

CAREERS IN MATHEMATICS

Careers in Mathematics

To the teacher:

- Ditto the appropriate handouts.
- Discuss the three categories — vocational-technical, college, job.
- Additional alternatives

  Select (circle) each occupation if you know someone who is in the occupation.

  Talk with someone who is in each of the occupations in which you are interested.

  Invite a representative of an occupation of interest to you to speak to your class.

  Select three occupations you might find interesting.

  Work in small groups based on common interests or work individually to find out about the occupations.

- Guidelines for students in seeking information about an occupation

  What are some advantages of the job?

  What are some disadvantages of the job?

  What are the job opportunities? (See *Occupational Handbook*)

  What are the income opportunities associated with the job?

  Describe the working conditions.

  How much formal education is needed?

  Will I need to take any more mathematics?

  Go to the media center, local or regional libraries to find filmstrips describing the selected occupation.

  Check resources in the counselor's office.

  Check resources in the cooperative program areas of your school.

  Visit the state employment offices (as well as private employment agencies) to find what future possible job opportunities might be available. Visit places of employment.

WILL MATHEMATICS BE A PART OF MY FUTURE?

Yes. Our society is using mathematics more and more. More doors are closed without it.

WHAT IS THE BEST WAY TO PREPARE FOR MY FUTURE WHILE I'M IN HIGH SCHOOL?

You can expect to change jobs about six times in your lifetime. You need to be ready. By your being in geometry, you've already taken a positive step in broadening the careers available to you.

CAN I GET A JOB AFTER HIGH SCHOOL WITHOUT TAKING ANY MORE MATHEMATICS?

Yes, however you may not advance as rapidly as someone with more mathematics.

DO I NEED MATHEMATICS TO PREPARE FOR COLLEGE?

Yes, for most programs in most colleges.

WHAT IF I DON'T KNOW WHAT I'M GOING TO DO AFTER HIGH SCHOOL?

athematics! Chances are you'll o, don't get caught without it.

WILL MATHEMATICS HELP ME GET PROMOTIONS IN MY JOB?

Most likely, yes. In many fields you must have mathematics to get promoted to higher levels.

WHAT IF I DON'T TAKE MATHEMATICS NEXT YEAR?

You will probably lose your chance to take all the high school mathematics you might need.

IF I NEED MORE MATHEMATICS WILL I BE ABLE TO TAKE IT AFTER HIGH SCHOOL?

Yes, but it could cause lengthy delays and unnecessary expense in your future education and job opportunities.

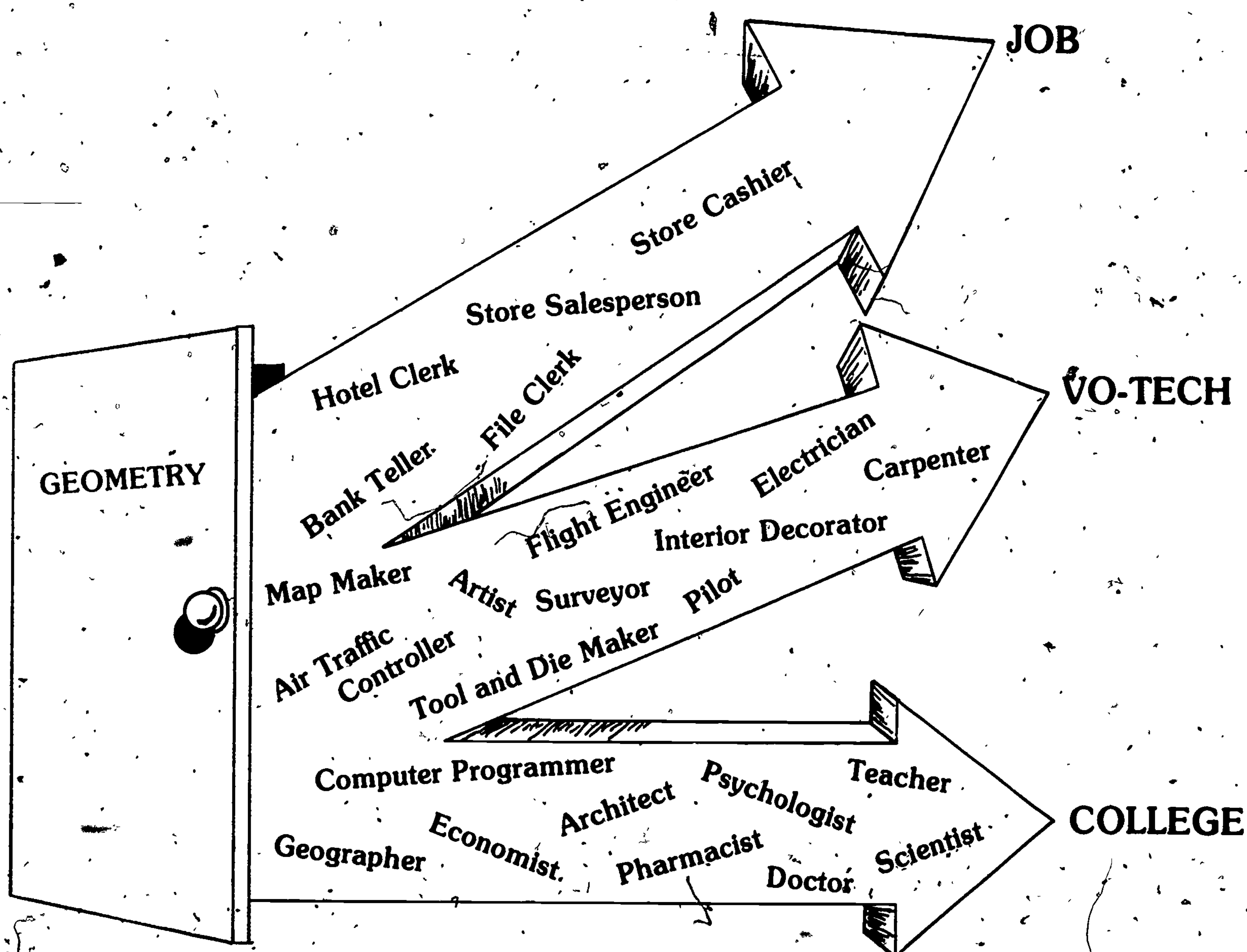
WHAT ARE THE FASTEST GROWING FIELDS?

Computer Science
Health and Energy Fields

HOW CAN I GET MORE INFORMATION ABOUT MATHEMATICS AND MY FUTURE?

Mathematics Teachers
Guidance Office
Media Center

Mathematics opens the doorways to effective living.



Georgia Department of Education
Atlanta, Georgia 30334
Charles McDaniel
e Superintendent of Schools
1981

WILL MATHEMATICS BE A PART OF MY FUTURE?

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WHAT IS THE BEST WAY TO PREPARE FOR MY FUTURE WHILE I'M IN HIGH SCHOOL?

You can expect to change jobs about six times in your lifetime. You need to be ready. By your being in analysis or probability and statistics you've already taken a positive step in broadening the careers available to you.

•

WITHOUT FURTHER FORMAL STUDIES WILL ANALYSIS HELP ME GET A JOB?

Yes. Although you may not advance as rapidly as someone with a college degree, you will still have the advantage when applicant selections or promotional considerations are made.

•

WHAT ARE THE FASTEST GROWING FIELDS?

- Computer Science
- Health and Energy Fields

FOR WHAT KINDS OF COLLEGE COURSES WOULD ANALYSIS BE ESPECIALLY HELPFUL?

For those courses that involve topics such as statistics, physics, economics, chemistry and engineering — higher level courses leading to professional careers.

•

CAN I EXPECT MY EARNINGS TO INCREASE AS A RESULT OF STUDYING ANALYSIS?

While money is important when selecting a job, the prime considerations should include

- job security
- job satisfaction
- opportunities for advancement
- employee benefits

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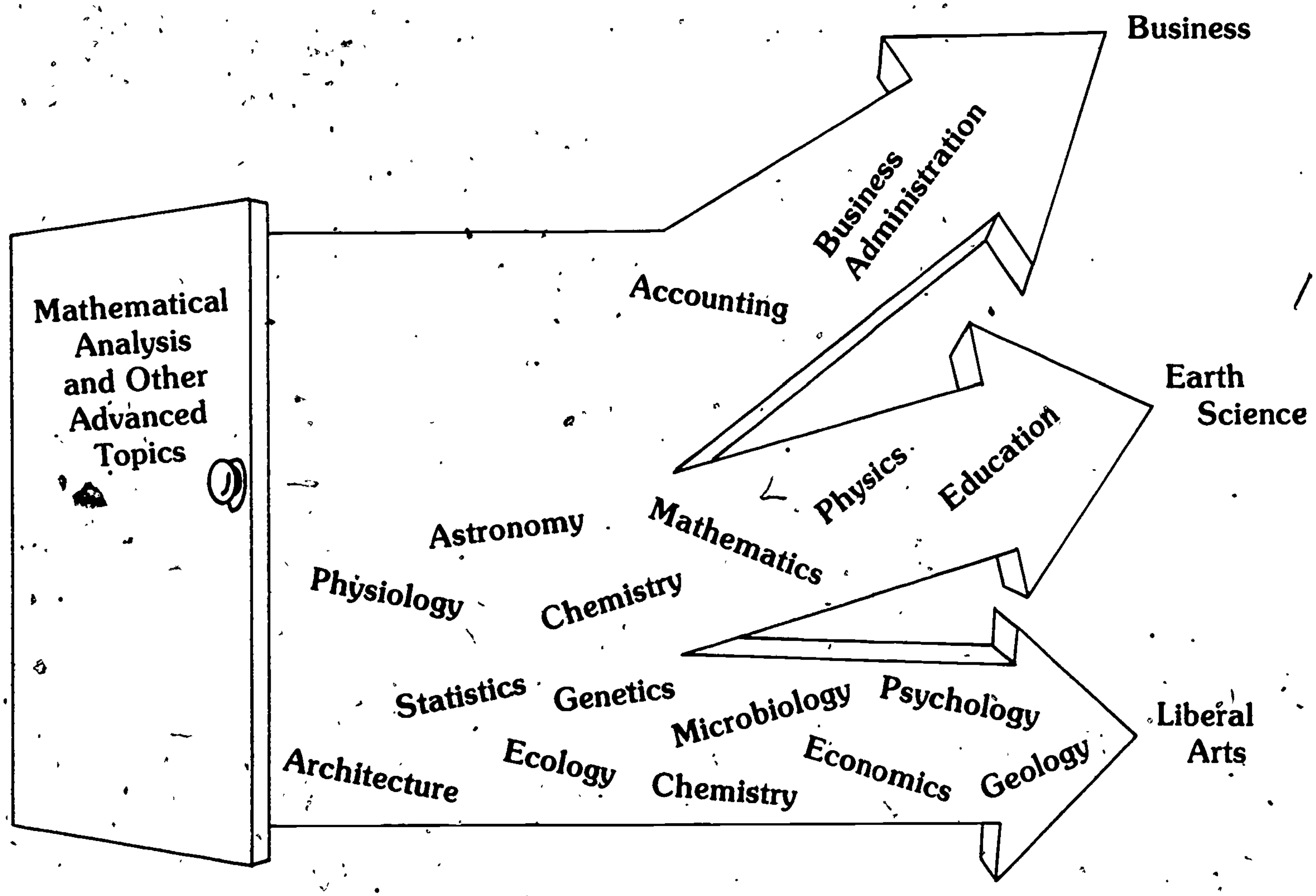
HOW CAN I GET MORE INFORMATION?

Mathematics teachers

Sources in guidance office, media center

Current edition of Occupational Outlook Handbook

Mathematics Opens Your Career Doors.



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Georgia Department of Education
Atlanta, Georgia 30334
Charles McDaniel
State Superintendent of Schools
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